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Anisotropic ball Campanato-type function spaces and their applications

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Abstract

Let A be a general expansive matrix and let X be a ball quasi-Banach function space on $\mathbb{R}^n$, whose certain power (namely its convexification) supports a Fefferman–Stein vector-valued maximal inequality and the associate space of whose other power supports the boundedness of the powered Hardy–Littlewood maximal operator. The authors first introduce some anisotropic ball Campanato-type function spaces associated with both A and X, prove that these spaces are dual spaces of anisotropic Hardy spaces $H_X^A(\mathbb{R}^n)$ associated with both A and X, and obtain various anisotropic Littlewood–Paley function characterizations of $H_X^A(\mathbb{R}^n)$. Also, as applications, the authors establish several equivalent characterizations of anisotropic ball Campanato-type function spaces, which, combined with the atomic decomposition of tent spaces associated with both A and X, further induce their Carleson measure characterization. All these results have a wide range of generality and, particularly, even when they are applied to Morrey spaces and Orlicz-slice spaces, some of the obtained results are also new. The novelties of this article are reflected in that, to overcome the essential difficulties caused by the absence of both an explicit expression and the absolute continuity of the quasi-norm $\|\cdot\|_X$ under consideration, the authors embed X under consideration

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into the anisotropic weighted Lebesgue space with certain special weight and then fully use the known results of this weighted Lebesgue space.

Keywords Expansive matrix · Ball quasi-Banach function space · Hardy space · Campanato-type function space · Duality · Littlewood–Paley function · Carleson measure

Mathematics Subject Classification Primary 46E30; Secondary 42B30 · 42B25 · 42B35 · 28C20

1 Introduction

Recall that the dual theory of classical Hardy spaces on the Euclidean space $\mathbb{R}^n$ plays an important role in many branches of analysis, such as harmonic analysis and partial differential equations, and has been systematically considered and developed so far; see, for instance, [29, 67]. Indeed, in 1969, Duren et al. [27] first identified the Lipshitz space with the dual space of the Hardy space $H^p(\mathbb{D})$ of holomorphic functions, where $p \in (0, 1)$ and the *symbol* $\mathbb{D}$ denotes the unit disc of $\mathbb{C}$. Later, Walsh [81] further extended this dual result to the Hardy space on the upper half-plane $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$. On the real Hardy spaces, the famous dual theorem, that is, the space $\mathrm{BMO}(\mathbb{R}^n)$ of functions with bounded mean oscillation is the dual space of the Hardy space $H^1(\mathbb{R}^n)$, is due to Fefferman and Stein [29]. Moreover, it is worth pointing out that the complete dual theory of the Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ was given by Taibleson and Weiss [75], in which the dual space of $H^p(\mathbb{R}^n)$ proves the special Campanato space introduced by Campanato [15].

Recently, Sawano et al. [69] originally introduced the ball quasi-Banach function space X which further generalizes the Banach function space in [3] in order to include weighted Lebesgue spaces, Morrey spaces, mixed-norm Lebesgue spaces, Orlicz-slice spaces, and Musielak–Orlicz spaces. Observe that the aforementioned several function spaces are not quasi-Banach function spaces which were originally introduced in [3, Definitions 1.1 and 1.3]; see, for instance, [69, 72, 83, 90]. In the same article [69], Sawano et al. also introduced the Hardy space $H_X(\mathbb{R}^n)$, associated with X, and established its various maximal function characterizations by assuming that the Hardy–Littlewood maximal operator is bounded on the p-convexification of X, as well as its several other real-variable characterizations, respectively, in terms of atoms, molecules, and Lusin area functions by assuming that the Hardy–Littlewood maximal operator satisfies a Fefferman–Stein vector-valued inequality on certain power (namely its convexification) of X and the powered Hardy–Littlewood maximal operator is bounded on the associate space of certain power of X.

Later, Wang et al. [82] further established the Littlewood–Paley g-function and the Littlewood–Paley g_λ^*-function characterizations of both $H_X(\mathbb{R}^n)$ and its local version $h_X(\mathbb{R}^n)$ and obtained the boundedness of Calderón–Zygmund operators and pseudo-differential operators, respectively, on $H_X(\mathbb{R}^n)$ and $h_X(\mathbb{R}^n)$; Yan et al. [87] established the dual theorem and the intrinsic square function characterizations of $H_X(\mathbb{R}^n)$; Zhang et al. [89] introduced some new ball Campanato-type function space which proves the dual space of $H_X(\mathbb{R}^n)$ and established its Carleson measure characterization. Very recently, on spaces $\mathcal{X}$ of homogeneous type, Yan et al. [85, 86] introduced ball quasi-Banach function spaces $Y(\mathcal{X})$ and Hardy-type spaces $H_Y(\mathcal{X})$, associated with $Y(\mathcal{X})$, and developed a complete real-variable theory of $H_Y(\mathcal{X})$. For more studies about ball quasi-Banach function spaces, we refer the reader to [16, 43, 44, 68, 74, 77, 88].

On the other hand, starting from 1970s, there has been an increasing interesting in extending classical function spaces arising in harmonic analysis from $\mathbb{R}^n$ to various anisotropic settings and some other underlying spaces; see, for instance, [21, 31–33, 35, 65, 73, 76, 78, 79]. The study of function spaces on $\mathbb{R}^n$ associated with anisotropic dilations was originally started from the celebrated articles [12–14] of Calderón and Torchinsky on anisotropic Hardy spaces. In 2003, Bownik [4] introduced and investigated the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ with $p \in (0, \infty)$, where A is a general expansive matrix on $\mathbb{R}^n$. Since then, various variants of classical Hardy spaces over the anisotropic Euclidean space have been introduced and their real-variable theories have been systematically developed. To be precise, Bownik et al. [7] further extended the anisotropic Hardy space to the weighted setting. Li et al. [50] introduced the anisotropic Musielak–Orlicz Hardy space $H_A^{\varphi}(\mathbb{R}^n)$, where φ is an anisotropic Musielak–Orlicz function, and characterized $H_A^{\varphi}(\mathbb{R}^n)$ by several maximal functions and atoms. Liu et al. [62, 64] first introduced the anisotropic Hardy–Lorentz space $H_A^{p,q}(\mathbb{R}^n)$, with $p \in (0, 1]$ and $q \in (0, \infty]$, and established their several real-variable characterizations, respectively, in terms of atoms or finite atoms, molecules, maximal functions, and Littlewood–Paley functions, which are further applied to obtain the real interpolation theorem of $H_A^{p,q}(\mathbb{R}^n)$ and the boundedness of anisotropic Calderón–Zygmund operators on $H_A^{p,q}(\mathbb{R}^n)$ including the critical case. Liu et al. [56, 60] and Huang et al. [39] further generalized the corresponding results in [62, 64] to variable Hardy spaces and mixed-norm Hardy spaces, respectively. Recently, Liu et al. [61, 63] introduced the anisotropic variable Hardy–Lorentz space $H_A^{p(\cdot),q}(\mathbb{R}^n)$, where $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ is a variable exponent function satisfying the globally log-Hölder continuous condition and $q \in (0, \infty]$, and developed a complete real-variable theory of these spaces including various equivalent characterizations and the boundedness of sublinear operators. Independently, Almeida et al. [1] also investigated the anisotropic variable Hardy–Lorentz space $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, where $p(\cdot)$ and $q(\cdot)$ are nonnegative measurable functions on $(0, \infty)$. In [1], equivalent characterizations of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ in terms of maximal functions and atoms were established. It is remarkable that the anisotropic variable Hardy–Lorentz space $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ in [1] and that in [61, 63] can not cover each other because the variable exponent $p(\cdot)$ in [1] is only defined on $(0, \infty)$, instead of $\mathbb{R}^n$. Particularly, Huang et al. [40, 41] further enriched the real-variable theory of anisotropic mixed-norm Campanato spaces and anisotropic variable Campanato spaces and established the dual theory of both anisotropic Hardy spaces $H_A^{\vec{p}}(\mathbb{R}^n)$ and $H_A^{p(\cdot)}(\mathbb{R}^n)$ with the full ranges of both $\vec{p}$ and $p(\cdot)$. For more studies about function spaces on the anisotropic Euclidean space, we refer the reader to [5, 8, 19, 20, 47, 48, 51].

Recall that the anisotropic Hardy space $H_X^A(\mathbb{R}^n)$ associated with both A and X was first introduced and studied by Wang et al. [84], in which they characterized $H_X^A(\mathbb{R}^n)$ in terms of maximal functions, atoms, finite atoms, and molecules and obtained the boundedness of the anisotropic Calderón–Zygmund operators on $H_X^A(\mathbb{R}^n)$. Motivated by this and [89], a quite *natural question* arises: can we prove whether or not the dual space of $H_X^A(\mathbb{R}^n)$ is the anisotropic ball Campanato-type function space and characterize this space by the Carleson measure? The main target of this article is to give an affirmative answer to this question. Indeed, to answer this question and also to enrich the real-variable theory of anisotropic Campanato spaces associated with both A and X, in this article, by borrowing some ideas from [89], namely considering finite linear combinations of atoms as a whole instead of a single atom, we introduce the anisotropic ball Campanato-type function space and give some applications. Using this and the additional assumptions that the Hardy–Littlewood maximal operator satisfies some Fefferman–Stein vector-valued inequality on certain power of X and the powered Hardy–Littlewood maximal operator is bounded on the associate space of certain power of X, we get rid of the dependence on the concavity of $\|\cdot\|_X$ and prove that the dual space of $H_X^A(\mathbb{R}^n)$ is just the anisotropic ball Campanato-type function space. From this, we further deduce several equivalent characterizations of anisotropic ball Campanato-type function spaces. Moreover, via embedding X into a certain anisotropic weighted Lebesgue space, we overcome the difficulty caused by the absence of both an explicit expression and the absolute continuity of the quasi-norm $\|\cdot\|_X$ under consideration and establish the anisotropic Littlewood–Paley characterizations of $H_X^A(\mathbb{R}^n)$, which, together with the dual theorem of $H_X^A(\mathbb{R}^n)$ and the atomic decomposition of anisotropic tent spaces associated with X, finally imply the Carleson measure characterization of anisotropic ball Campanato-type function spaces.

It is remarkable that the results obtained in this article have a wide range of generality because ball quasi-Banach function spaces include so many important function spaces. Particularly, when X becomes the Morrey space, the Littlewood–Paley function characterizations of anisotropic Hardy–Morrey spaces are new while the dual theorem and the Carleson measure characterization are not applicable because Morrey spaces do not have any absolutely continuous quasi-norm; when X becomes the Orlicz-slice space, the obtained results are completely new; when X becomes the weighted Lebesgue space or the Orlicz space, the dual theorem and the Carleson measure characterization are new while the Littlewood–Paley function characterizations of anisotropic Hardy-type spaces were obtained in [49]; when X becomes the Lorentz space, the variable Lebesgue space, or the mixed-norm Lebesgue space, the obtained results coincide with those in [39–41, 59, 60, 64]. Obviously, due to the flexibility and the operability, more applications of these results obtained in this article to newfound function spaces are completely possible.

The remainder of this article is organized as follows.

In Sect. 2, we recall some notation and concepts which are used throughout this article. More precisely, we first recall the definitions of the expansive matrix A, the step homogeneous quasi-norm ρ, and the ball quasi-Banach function space X. Then we make some mild assumptions on the boundedness of the (powered) Hardy–Littlewood maximal operator on both certain power of X and its associate space, which are needed throughout this article. Finally, we recall the concept of the non-tangential (grand) maximal function.

The aim of Sect. 3 is to give the dual space of the anisotropic Hardy space $H_X^A(\mathbb{R}^n)$ (see Theorem 3.15 below). To this end, we first introduce the anisotropic ball Campanato-type function space $\mathcal{L}^{A}_{X,q,d,s}(\mathbb{R}^n)$ (see Definition 3.3 below) and give an equivalent quasi-norm characterization of $\mathcal{L}^{A}_{X,q,d,s}(\mathbb{R}^n)$ (see Proposition 3.5 below). Using these, both the known atomic and the known finite atomic characterizations of $H^A_X(\mathbb{R}^n)$, and the assumptions that the Hardy–Littlewood maximal operator satisfies some Fefferman–Stein vector-valued inequality on certain power of X and the powered Hardy–Littlewood maximal operator is bounded on the associate space of certain power of X, we prove that the dual space of $H^A_X(\mathbb{R}^n)$ is just $\mathcal{L}^{A}_{X,q',d,\theta_0}(\mathbb{R}^n)$. At the end of this section, we also give its invariance of $\mathcal{L}^{A}_{X,q,d,s}(\mathbb{R}^n)$ on different indices q and d; see Corollary 3.16 below.

In Sect. 4, applying the dual result obtained in Theorem 3.15 and a key estimate (see Lemma 4.2 below), we obtain several equivalent characterizations of $\mathcal{L}^{A}_{X,q,d,\theta_0}(\mathbb{R}^n)$ (see Theorems 4.1 and 4.3 below), which are further applied to establish the Carleson measure characterization of $\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)$ in Sect. 6.

Section 5 is devoted to establishing the anisotropic Littlewood–Paley function characterization of $H^A_X(\mathbb{R}^n)$, including the anisotropic Lusin area function, the anisotropic Littlewood–Paley g-function, and the anisotropic Littlewood–Paley g^*_λ-function, respectively, in Theorems 5.4, 5.5, and 5.6 below. We first prove Theorem 5.4. To this end, we first show that the quasi-norms in X of the anisotropic Lusin area functions defined by different Schwartz functions are equivalent (see Theorem 5.7 below). Then, via borrowing some ideas from [64] and the anisotropic Calderón reproducing formula (see Lemma 5.2 below), we complete the proof of Theorem 5.4. From this and an approach initiated by Ullrich [80] and further developed by Liang et al. [55] and Liu et al. [61], together with Fefferman–Stein vector-valued inequality on certain power of X, we obtain the anisotropic Littlewood–Paley g-function and the anisotropic Littlewood–Paley g^*_λ-function characterizations.

In Sect. 6, we establish the Carleson measure characterization of $\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)$ (see Theorem 6.3 below). Indeed, via using Theorems 3.15, 4.1, and 5.4, as well as the atomic decomposition of anisotropic tent spaces associated with X (see Lemma 6.7 below), we show that a measurable function h belongs to $\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)$ if and only if h generates an X–Carleson measure $d\mu$. Moreover, the norm of the X–Carleson measure $d\mu$ is equivalent to the $\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)$-norm of h.

In Sect. 7, we apply all the main results obtained in the above sections to several specific ball quasi-Banach function spaces. Particularly, the results about Morrey spaces and Orlicz-slice spaces are completely new and stated, respectively, in Sects. 7.1 and 7.2; part of the results about weighted Lebesgue spaces and Orlicz spaces obtained in this article are new and stated, respectively, in Sects. 7.6 and 7.7; the results about Lorentz spaces, variable Lebesgue spaces, and mixed-norm Lebesgue spaces coincide with the known ones, which are successively stated in Sects. 7.3, 7.4, and 7.5.

At the end of this section, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$, $\mathbb{Z}^n_+ := (\mathbb{Z}_+)^n$, and $\mathbf{0}$ be the *origin* of $\mathbb{R}^n$. For any multi-index $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ and any $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$, and $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We denote by C a *positive constant* which is independent of the main parameters, but may vary from line to line. We use $C_{(\alpha,\ldots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \ldots$. The symbol $f \lesssim g$ means $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. If

$f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g = h$ or $f \lesssim g \le h$. For any $q \in [1, \infty]$, we denote by q' its *conjugate index*, that is, $1/q + 1/q' = 1$. For any $x \in \mathbb{R}^n$, we denote by $|x|$ the n-dimensional *Euclidean metric* of x. If E is a subset of $\mathbb{R}^n$, we denote by $\mathbf{1}_E$ its *characteristic function* and by $E^\complement$ the set $\mathbb{R}^n \setminus E$. For any $r \in (0, \infty)$ and $x \in \mathbb{R}^n$, we denote by $B(x, r)$ the ball centered at x with the radius r, that is, $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. For any ball B, we use x_B to denote its center and r_B its radius, and denote by λB for any $\lambda \in (0, \infty)$ the ball concentric with B having the radius λr_B. We also use $\epsilon \to 0^+$ to denote $\epsilon \in (0, \infty)$ and $\epsilon \to 0$. Let X and Y be two normed vector spaces, respectively, with the norm $\|\cdot\|_X$ and the norm $\|\cdot\|_Y$; then we use $X \hookrightarrow Y$ to denote $X \subset Y$ and there exists a positive constant C such that, for any $f \in X$, $\|f\|_Y \le C\|f\|_X$. For any measurable function f on $\mathbb{R}^n$ and any measurable set $E \subset \mathbb{R}^n$ with $|E| \in (0, \infty)$, let

$$\fint_E f(x)\,dx := \frac{1}{|E|}\int_E f(x)\,dx.$$

At last, when we prove a theorem or the like, we always use the same symbols in the wanted proved theorem or the like.

2 Preliminaries

In this section, we first recall some notation and concepts on dilations (see, for instance [4, 36]) as well as ball quasi-Banach function spaces (see, for instance, [69, 82, 83, 87, 90]). We begin with recalling the concept of the expansive matrix from [4].

Definition 2.1 A real $n \times n$ matrix A is called an *expansive matrix* (shortly, a *dilation*) if

$$\min_{\lambda \in \sigma(A)} |\lambda| > 1,$$

here and thereafter, $\sigma(A)$ denotes the *set of all eigenvalues of* A.

Let $A := (a_{i,j})_{1 \le i,j \le n}$ be a dilation. Then let

$$b := |\det A|, \tag{2.1}$$

where $\det A$ denotes the *determinant* of A, and define the *matrix norm* $\|A\|$ by setting

$$\|A\| := \left(\sum_{i,j=1}^{n} |a_{i,j}|^2\right)^{1/2}. \tag{2.2}$$

Then it follows from [4, p. 6, (2.7)] that $b \in (1, \infty)$. By the fact that there exists an open and symmetry ellipsoid Δ, with $|\Delta| = 1$, and an $r \in (1, \infty)$ such that $\Delta \subset r\Delta \subset A\Delta$ (see [4, p. 5, Lemma 2.2]), we find that, for any $k \in \mathbb{Z}$,

$$B_k := A^k \Delta \tag{2.3}$$

is open, $B_k \subset rB_k \subset B_{k+1}$, and $|B_k| = b^k$. For any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, the ellipsoid $x + B_k$ is called a *dilated ball*. In what follows, we always let $\mathcal{B}$ be the set of all such dilated balls, that is,

$$\mathcal{B} := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\} \tag{2.4}$$

and let

$$\tau := \inf\left\{l \in \mathbb{Z} : r^l \geq 2\right\}. \tag{2.5}$$

Let $\lambda_-, \lambda_+ \in (0, \infty)$ satisfy that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+. \tag{2.6}$$

We point out that, if A is diagonalizable over $\mathbb{R}$, then we may let

$$\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\} \text{ and } \lambda_+ := \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

Otherwise, we may choose them sufficiently close to these equalities in accordance with what we need in our arguments.

The following definition of the homogeneous quasi-norm is just [4, p. 6, Definition 2.3].

Definition 2.2 A *homogeneous quasi-norm*, associated with a dilation A, is a measurable mapping $\varrho : \mathbb{R}^n \to [0, \infty)$ such that

(i) $\varrho(x) = 0 \Longleftrightarrow x = \mathbf{0}$, where $\mathbf{0}$ denotes the origin of $\mathbb{R}^n$;
(ii) $\varrho(Ax) = b\varrho(x)$ for any $x \in \mathbb{R}^n$;
(iii) there exists an $A_0 \in [1, \infty)$ such that, for any $x, y \in \mathbb{R}^n$,

$$\varrho(x + y) \leq A_0\,[\varrho(x) + \varrho(y)].$$

In the standard Euclidean space case, let $A := 2\,I_{n\times n}$ and, for any $x \in \mathbb{R}^n$, $\varrho(x) := |x|^n$. Then ϱ is an example of homogeneous quasi-norms associated with A on $\mathbb{R}^n$. Here and thereafter, $I_{n\times n}$ always denotes the $n \times n$ *unit matrix* and $|\cdot|$ the *Euclidean norm* in $\mathbb{R}^n$.

For a fixed dilation A, by [4, p. 6, Lemma 2.4], we define the following quasi-norm which is used throughout this article.

Definition 2.3 Define the *step homogeneous quasi-norm* ρ on $\mathbb{R}^n$, associated with the dilation A, by setting

$$\rho(x) := \begin{cases} b^k & \text{if } x \in B_{k+1} \setminus B_k, \\ 0 & \text{if } x = \mathbf{0}, \end{cases}$$

where b is the same as in (2.1) and, for any $k \in \mathbb{Z}$, B_k the same as in (2.3).

Then $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [23], where dx denotes the n-dimensional Lebesgue measure. For more studies on the real-variable theory of function spaces over spaces of homogeneous type, we refer the reader to [9–11, 52–54].

Throughout this article, we always let A be a dilation in Definition 2.1, b the same as in (2.1), ρ the step homogeneous quasi-norm in Definition 2.3, $\mathcal{B}$ the set of all dilated balls in (2.4), $\mathscr{M}(\mathbb{R}^n)$ the *set of all measurable functions* on $\mathbb{R}^n$, and B_k for any $k \in \mathbb{Z}$ the same as in (2.3). Now, we recall the definition of ball quasi-norm Banach function spaces (see [69]).

Definition 2.4 A quasi-normed linear space $X \subset \mathscr{M}(\mathbb{R}^n)$, equipped with a quasi-norm $\|\cdot\|$ which makes sense for the whole $\mathscr{M}(\mathbb{R}^n)$, is called a *ball quasi-Banach function space* if it satisfies

(i) for any $f \in \mathscr{M}(\mathbb{R}^n)$, $\|f\|_X = 0$ implies that $f = 0$ almost everywhere;
(ii) for any $f, g \in \mathscr{M}(\mathbb{R}^n)$, $|g| \le |f|$ almost everywhere implies that $\|g\|_X \le \|f\|_X$;
(iii) for any $\{f_m\}_{m\in\mathbb{N}} \subset \mathscr{M}(\mathbb{R}^n)$ and $f \in \mathscr{M}(\mathbb{R}^n)$, $0 \le f_m \uparrow f$ as $m \to \infty$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$ as $m \to \infty$;
(iv) $\mathbf{1}_B \in X$ for any dilated ball $B \in \mathcal{B}$.

Moreover, a ball quasi-Banach function space X is called a *ball Banach function space* if it satisfies:

(v) for any $f, g \in X$, $\|f + g\|_X \le \|f\|_X + \|g\|_X$;
(vi) for any given dilated ball $B \in \mathcal{B}$, there exists a positive constant $C_{(B)}$ such that, for any $f \in X$,

$$\int_B |f(x)|\,dx \le C_{(B)}\|f\|_X.$$

Remark 2.5 (i) As was mentioned in [84, Remark 2.5(i)], if $f \in \mathscr{M}(\mathbb{R}^n)$, then $\|f\|_X = 0$ if and only if $f = 0$ almost everywhere; if $f, g \in \mathscr{M}(\mathbb{R}^n)$ and $f = g$ almost everywhere, then $\|f\|_X \sim \|g\|_X$ with the positive equivalence constants independent of both f and g.
(ii) As was mentioned in [84, Remark 2.5(ii)], if we replace any dilated ball $B \in \mathcal{B}$ in Definition 2.4 by any bounded measurable set E or by any ball $B(x, r)$ with $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, we obtain its another equivalent formulation.
(iii) By [26, Theorem 2], we find that both (ii) and (iii) of Definition 2.4 imply that any ball quasi-Banach function space is complete.

Now, we recall the concepts of the p-convexification and the concavity of ball quasi-Banach function spaces, which is a part of [69, Definition 2.6].

Definition 2.6 Let X be a ball quasi-Banach function space and $p \in (0, \infty)$.

(i) The *p-convexification* X^p of X is defined by setting

$$X^p := \left\{f \in \mathscr{M}(\mathbb{R}^n) : \ |f|^p \in X\right\}$$

equipped with the *quasi-norm* $\|f\|_{X^p} := \||f|^p\|_X^{1/p}$ for any $f \in X^p$.

(ii) The space X is said to be *concave* if there exists a positive constant C such that, for any $\{f_k\}_{k\in\mathbb{N}} \subset \mathscr{M}(\mathbb{R}^n)$,

$$\sum_{k=1}^{\infty} \|f_k\|_X \le C \left\| \sum_{k=1}^{\infty} |f_k| \right\|_X .$$

In particular, when $C = 1$, X is said to be *strictly concave*.

The associate space X' of any given ball Banach function space X is defined as follows; see [3, Chapter 1, Sect. 2] or [69, p. 9].

Definition 2.7 For any given ball Banach function space X, its *associate space* (also called the *Köthe dual space*) X' is defined by setting

$$X' := \left\{ f \in \mathscr{M}(\mathbb{R}^n) : \ \|f\|_{X'} := \sup_{g\in X,\ \|g\|_X=1} \|fg\|_{L^1(\mathbb{R}^n)} < \infty \right\},$$

where $\|\cdot\|_{X'}$ is called the *associate norm* of $\|\cdot\|_X$.

Remark 2.8 From [69, Proposition 2.3], we deduce that, if X is a ball Banach function space, then its associate space X' is also a ball Banach function space.

Next, we recall the concept of absolutely continuous quasi-norms of X as follows (see [82, Definition 3.2] for the classical Euclidean space case and [85, Definition 6.1] for the case of spaces of homogeneous type).

Definition 2.9 Let X be a ball quasi-Banach function space. A function $f \in X$ is said to have an *absolutely continuous quasi-norm* in X if $\|f\mathbf{1}_{E_j}\|_X \downarrow 0$ whenever $\{E_j\}_{j=1}^{\infty}$ is a sequence of measurable sets satisfying $E_j \supset E_{j+1}$ for any $j \in \mathbb{N}$ and $\bigcap_{j=1}^{\infty} E_j = \emptyset$. Moreover, X is said to have an *absolutely continuous quasi-norm* if, for any $f \in X$, f has an absolutely continuous quasi-norm in X.

Now, we recall the concept of the Hardy–Littlewood maximal operator. Let $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ denote the *set of all locally integrable functions* on $\mathbb{R}^n$. Recall that the *Hardy–Littlewood maximal operator* $\mathcal{M}(f)$ of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{k\in\mathbb{Z}} \sup_{y\in x+B_k} \fint_{y+B_k} |f(z)|\,dz = \sup_{x\in B\in\mathcal{B}} \fint_B |f(z)|\,dz,$$

where $\mathcal{B}$ is the same as in (2.4) and the last supremum is taken over all balls $B \in \mathcal{B}$. For any given $\alpha \in (0, \infty)$, the *powered Hardy–Littlewood maximal operator* $\mathcal{M}^{(\alpha)}$ is defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}^{(\alpha)}(f)(x) := \left\{\mathcal{M}\left(|f|^{\alpha}\right)(x)\right\}^{\frac{1}{\alpha}} .$$

Throughout this article, we also need the following two fundamental assumptions about the boundedness of $\mathcal{M}$ on the $\frac{1}{p}$-convexification $X^{\frac{1}{p}}$ of the given ball quasi-Banach function space X and the boundedness of a certain powered Hardy–Littlewood maximal operator on the associate space of the $\frac{1}{\theta_0}$-convexification of X.

Assumption 2.10 Let X be a ball quasi-Banach function space. Assume that there exists a $p_- \in (0, \infty)$ such that, for any $p \in (0, p_-)$ and $u \in (1, \infty)$, there exists a positive constant C, depending on both p and u, such that, for any $\{f_k\}_{k=1}^{\infty} \subset \mathscr{M}(\mathbb{R}^n)$,

$$\left\| \left\{ \sum_{k=1}^{\infty} [\mathcal{M}(f_k)]^u \right\}^{\frac{1}{u}} \right\|_{X^{\frac{1}{p}}} \leq C \left\| \left\{ \sum_{k=1}^{\infty} |f_k|^u \right\}^{\frac{1}{u}} \right\|_{X^{\frac{1}{p}}}.$$

Remark 2.11 Let X be a ball-Banach function space. Suppose that $\mathcal{M}$ is bounded on both X and X'. By an argument similar to that used in the proof of [24, Theorem 4.10], we find that $\mathcal{M}$ satisfies Assumption 2.10 with $p_- = 1$.

In what follows, for any given $p_- \in (0, \infty)$, we always let

$$\underline{p} := \min\{p_-, 1\}. \tag{2.7}$$

Assumption 2.12 Let $p_- \in (0, \infty)$ and X be a ball quasi-Banach function space. Assume that there exists a $\theta_0 \in (0, \underline{p})$, with $\underline{p}$ the same as in (2.7), and a $p_0 \in (\theta_0, \infty)$ such that X^{1/θ_0} is a ball Banach function space and, for any $f \in (X^{1/\theta_0})'$,

$$\left\| \mathcal{M}^{((p_0/\theta_0)')}(f) \right\|_{(X^{1/\theta_0})'} \leq C \|f\|_{(X^{1/\theta_0})'},$$

where C is a positive constant, independent of f, and $\frac{1}{p_0/\theta_0} + \frac{1}{(p_0/\theta_0)'} = 1$.

Next, recall that a *Schwartz function* is a function $\varphi \in C^{\infty}(\mathbb{R}^n)$ satisfying that, for any $k \in \mathbb{Z}_+$ and any multi-index $\alpha \in \mathbb{Z}_+^n$,

$$\|\varphi\|_{\alpha,k} := \sup_{x \in \mathbb{R}^n} [\rho(x)]^k |\partial^{\alpha} \varphi(x)| < \infty.$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ the *set of all Schwartz functions*, equipped with the topology determined by $\{\|\cdot\|_{\alpha,k}\}_{\alpha \in \mathbb{Z}_+^n, k \in \mathbb{Z}_+}$. Then $\mathcal{S}'(\mathbb{R}^n)$ is defined to be the *dual space* of $\mathcal{S}(\mathbb{R}^n)$, equipped with the weak-$*$ topology. For any $N \in \mathbb{Z}_+$, let

$$\mathcal{S}_N(\mathbb{R}^n) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_{\alpha,k} \leq 1, |\alpha| \leq N, k \leq N \right\},$$

equivalently,

$$\varphi \in \mathcal{S}_N(\mathbb{R}^n) \Longleftrightarrow \|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)} := \sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n} \max\{1, [\rho(x)]^N\}|\partial^\alpha \varphi(x)| \le 1.$$

In what follows, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{Z}$, let $\varphi_k(\cdot) := b^{-k}\varphi(A^{-k}\cdot)$.

Definition 2.13 Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The *non-tangential maximal function* $M_\varphi(f)$ with respect to φ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\varphi(f)(x) := \sup_{k\in\mathbb{Z},\, y\in x+B_k} |f * \varphi_k(y)|.$$

Moreover, for any given $N \in \mathbb{N}$, the *non-tangential grand maximal function* $M_N(f)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N(f)(x) := \sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)} M_\varphi(f)(x). \tag{2.8}$$

3 Duality between $H_X^A(\mathbb{R}^n)$ and $\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$

In this section, we provide a description of the dual space of the anisotropic Hardy space $H_X^A(\mathbb{R}^n)$ associated with ball quasi-Banach function space X. This description is a consequence of the definition of the anisotropic ball Campanato-type function space, the atomic and the finite atomic characterizations of $H_X^A(\mathbb{R}^n)$ from [84], as well as some basic tools from functional analysis. To state the dual theorem, we first present the definition of $H_X^A(\mathbb{R}^n)$ from [84] as follows. In what follows, for any $\alpha \in \mathbb{R}$, we denote by $\lfloor\alpha\rfloor$ the largest integer not greater than α.

Definition 3.1 Let A be a dilation and X a ball quasi-Banach function space satisfying both Assumption 2.10 with $p_- \in (0,\infty)$ and Assumption 2.12 with the same p_-, $\theta_0 \in (0, \underline{p})$, and $p_0 \in (\theta_0, \infty)$, where $\underline{p}$ is the same as in (2.7). Assume that

$$N \in \mathbb{N} \cap \left[\left\lfloor \left(\frac{1}{\theta_0} - 1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor + 2, \infty\right). \tag{3.1}$$

The *anisotropic Hardy space* $H^A_{X,N}(\mathbb{R}^n)$, associated with both A and X, is defined by setting

$$H^A_{X,N}(\mathbb{R}^n) := \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|M_N(f)\|_X < \infty\right\},$$

where $M_N(f)$ is the same as in (2.8). Moreover, for any $f \in H^A_{X,N}(\mathbb{R}^n)$, let

$$\|f\|_{H^A_{X,N}(\mathbb{R}^n)} := \|M_N(f)\|_X.$$

Let A be a dilation and X the same as in Definition 3.1. In the remainder of this article, we always let

$$N_{X,\,A} := \left\lfloor \left(\frac{1}{\theta_0} - 1\right) \frac{\ln b}{\ln(\lambda_-)} \right\rfloor + 2. \tag{3.2}$$

Remark 3.2 (i) As was mentioned in [84, Remark 2.17(i)], the space $H^A_{X,\,N}(\mathbb{R}^n)$ is independent of the choice of N as long as $N \in \mathbb{N} \cap [N_{X,\,A}, \infty)$. In what follows, when $N \in \mathbb{N} \cap [N_{X,\,A}, \infty)$, we write $H^A_{X,\,N}(\mathbb{R}^n)$ simply by $H^A_X(\mathbb{R}^n)$.

(ii) We point out that, if $A := 2\, I_{n\times n}$, then $H^A_X(\mathbb{R}^n)$ coincides with $H_X(\mathbb{R}^n)$ which was introduced by Sawano et al. in [69].

In what follows, for any $d \in \mathbb{Z}_+$, $\mathcal{P}_d(\mathbb{R}^n)$ denotes the set of all the polynomials on $\mathbb{R}^n$ with degree not greater than d; for any ball $B \in \mathcal{B}$ and any locally integrable function g on $\mathbb{R}^n$, we use $P^d_B g$ to denote the *minimizing polynomial* of g with degree not greater than d, which means that $P^d_B g$ is the unique polynomial $f \in \mathcal{P}_d(\mathbb{R}^n)$ such that, for any $h \in \mathcal{P}_d(\mathbb{R}^n)$,

$$\int_B [g(x) - f(x)]h(x)\,dx = 0.$$

Next, we introduce the anisotropic ball Campanato-type function space associated with the ball quasi-Banach function space. In what follows, we use $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ to denote the *set of all q-order locally integrable functions* on $\mathbb{R}^n$.

Definition 3.3 Let A be a dilation, X a ball quasi-Banach function space, $q \in [1, \infty)$, $d \in \mathbb{Z}_+$, and $s \in (0, \infty)$. Then the *anisotropic ball Campanato-type function space* $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$, associated with X, is defined to be the set of all the $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\begin{aligned}\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} := \sup &\left\| \left\{ \sum_{i=1}^m \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \\ &\times \sum_{j=1}^m \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} \left| f(x) - P^d_{B^{(j)}} f(x) \right|^q \, dx \right]^{\frac{1}{q}}\end{aligned}$$

is finite, where the supremum is taken over all $m \in \mathbb{N}$, $\{B^{(j)}\}_{j=1}^m \subset \mathcal{B}$, and $\{\lambda_j\}_{j=1}^m \subset (0, \infty)$.

Remark 3.4 Let A, X, q, d, and s be the same as in Definition 3.3.

(i) If we have the basic assumption that $\|\{\sum_{i=1}^m [\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}]^s \mathbf{1}_{B^{(i)}}\}^{\frac{1}{s}}\|_X^{-1} \in (0, \infty)$, the index m in Definition 3.3 can be chosen as ∞; see Proposition 3.5 below.

(ii) Obviously, $\mathcal{P}_d(\mathbb{R}^n) \subset \mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$. Indeed, $\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} = 0$ if and only if $f \in \mathcal{P}_d(\mathbb{R}^n)$. Throughout this article, we always identify $f \in \mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ with $\{f + P : \ P \in \mathcal{P}_d(\mathbb{R}^n)\}$.

(iii) For any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$\||f\||_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} := \sup\inf \left\| \left\{ \sum_{i=1}^m \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \times \sum_{j=1}^m \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} |f(x) - P(x)|^q \, dx \right]^{\frac{1}{q}},$$

where the supremum is taken the same way as in Definition 3.3 and the infimum is taken over all the $P \in \mathcal{P}_d(\mathbb{R}^n)$. Then, similarly to the proof of [87, Lemma 2.6] with using [4, p. 49, (8.9)] instead of [87, Lemma 2.5], we easily find that $\||\cdot\||_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}$ is an equivalent quasi-norm of $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$.

Moreover, for the anisotropic ball Campanato-type function space $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$, we have the following equivalent quasi-norm.

Proposition 3.5 *Let A, X, q, d, and s be the same as in Definition 3.3. For any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, define*

$$\widetilde{\|f\|}_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} := \sup \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \times \sum_{j\in\mathbb{N}} \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} \left| f(x) - P^d_{B^{(j)}} f(x) \right|^q \, dx \right]^{\frac{1}{q}},$$

where the supremum is taken over all $\{B^{(j)}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ satisfying that

$$\left\| \left\{ \sum_{j\in\mathbb{N}} \left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right]^s \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{s}} \right\|_X \in (0,\infty). \tag{3.3}$$

Then, for any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\widetilde{\|f\|}_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} = \|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}.$$

Proof Let $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$. Obviously,

$$\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} \le \widetilde{\|f\|}_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}. \tag{3.4}$$

Conversely, let $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}}\subset(0,\infty)$ satisfy (3.3). From Definition 2.4(iii), it follows that

$$\begin{aligned}
&\lim_{m\to\infty}\left\|\left\{\sum_{i=1}^{m}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\\
&\quad\times\sum_{j=1}^{m}\frac{\lambda_j|B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\fint_{B^{(j)}}\left|f(x)-P^d_{B^{(j)}}f(x)\right|^q\,dx\right]^{\frac{1}{q}}\\
&=\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\\
&\quad\times\sum_{j\in\mathbb{N}}\frac{\lambda_j|B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\fint_{B^{(j)}}\left|f(x)-P^d_{B^{(j)}}f(x)\right|^q\,dx\right]^{\frac{1}{q}}.
\end{aligned}$$

Therefore, for any $\varepsilon\in(0,\infty)$, there exists an $m_0\in\mathbb{N}$ such that

$$\begin{aligned}
&\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\\
&\quad\times\sum_{j\in\mathbb{N}}\frac{\lambda_j|B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\fint_{B^{(j)}}\left|f(x)-P^d_{B^{(j)}}f(x)\right|^q\,dx\right]^{\frac{1}{q}}\\
&<\left\|\left\{\sum_{i=1}^{m_0}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\\
&\quad\times\sum_{i=1}^{m_0}\frac{\lambda_j|B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\fint_{B^{(j)}}\left|f(x)-P^d_{B^{(j)}}f(x)\right|^q\,dx\right]^{\frac{1}{q}}+\varepsilon\\
&\le\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}+\varepsilon,
\end{aligned}$$

which, together with the arbitrariness of both $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}}\subset(0,\infty)$ satisfying (3.3) and $\varepsilon\in(0,\infty)$, further implies that

$$\widetilde{\|f\|}_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}\le\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}.$$

This, combined with (3.4), then finishes the proof of Proposition 3.5. □

Now, we introduce another anisotropic ball Campanato-type function space $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$ associated with the ball quasi-Banach function space X.

Definition 3.6 Let A be a dilation, X a ball quasi-Banach function space, $q \in [1, \infty)$, and $d \in \mathbb{Z}_+$. Then the *Campanato space* $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$, associated with both A and X, is defined to be the set of all the $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)} := \sup_{B \in \mathcal{B}} \frac{|B|}{\|\mathbf{1}_B\|_X} \left\{ \fint_B \left| f(x) - P^d_B f(x) \right|^q \, dx \right\}^{\frac{1}{q}} < \infty,$$

where the supremum is taken over all the balls $B \in \mathcal{B}$ and $P^d_B f$ denotes the minimizing polynomial of f with degree not greater than d.

Remark 3.7 Let A, X, q, d, and s be the same as in Definition 3.3.

(i) From Definitions 3.3 and 3.6, it immediately follows that $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n) \subset \mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$ and this inclusion is continuous.

(ii) For any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$|||f|||_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)} := \sup_{B \in \mathcal{B}} \inf_{P \in \mathcal{P}_d(\mathbb{R}^n)} \frac{|B|}{\|\mathbf{1}_B\|_X} \left[\fint_B |f(x) - P(x)|^q \, dx \right]^{\frac{1}{q}}.$$

Then, similarly to [87, Lemma 2.6], we find that $|||\cdot|||_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)}$ is an equivalent quasi-norm of $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$.

Now, we give a basic inequality which is used throughout this article.

Lemma 3.8 *Let* $\{a_i\}_{i \in \mathbb{N}} \subset [0, \infty)$. *If* $\alpha \in [1, \infty)$, *then*

$$\left(\sum_{i \in \mathbb{N}} a_i \right)^\alpha \geq \sum_{i \in \mathbb{N}} a_i^\alpha.$$

The following proposition shows that, if the ball quasi-Banach function space X is concave and $s \in (0, 1]$, then the space $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ coincides with $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$ introduced in Definition 3.6.

Proposition 3.9 *Let X be a concave ball quasi-Banach function space,* $q \in [1, \infty)$, $d \in \mathbb{Z}_+$, *and* $s \in (0, 1]$. *Then*

$$\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n) = \mathcal{L}^A_{X,q,d}(\mathbb{R}^n) \tag{3.5}$$

with equivalent quasi-norms.

Proof We first show that

$$\mathcal{L}^A_{X,q,d}(\mathbb{R}^n) \subset \mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n) \tag{3.6}$$

and the inclusion is continuous. For this purpose, let $f \in \mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$. Then, from the assumption that X is concave, Definitions 2.4(ii) and 2.6(ii), and $s \in (0, 1]$, together with Lemma 3.8, we deduce that

$$\begin{aligned}\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} &\lesssim \sup \left(\sum_{i=1}^m \lambda_i\right)^{-1} \sum_{j=1}^m \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \\ &\quad \times \left[\fint_{B^{(j)}} \left| f(x) - P^d_{B^{(j)}} f(x) \right|^q \, dx \right]^{\frac{1}{q}} \\ &\le \sup \left(\sum_{i=1}^m \lambda_i\right)^{-1} \sum_{j=1}^m \lambda_j \|f\|_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)} = \|f\|_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)},\end{aligned}$$

where the supremum is taken over all $m \in \mathbb{N}$, $\{B^{(j)}\}_{j=1}^m \subset \mathcal{B}$, and $\{\lambda_j\}_{j=1}^m \subset (0, \infty)$. This further implies (3.6). Combining (3.6) and Remark 3.7(i), we obtain (3.5), which completes the proof of Proposition 3.9. □

Next, we establish the relation between $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ and $H^A_X(\mathbb{R}^n)$. To this end, we first recall the definitions of the anisotropic (X, q, d)-atom and the anisotropic finite atomic Hardy space $H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$ from [84, Definitions 4.1 and 5.1].

Definition 3.10 Let A be a dilation and X a ball quasi-Banach function space satisfying both Assumption 2.10 with $p_- \in (0, \infty)$ and Assumption 2.12 with the same p_-, $\theta_0 \in (0, \underline{p})$, and $p_0 \in (\theta_0, \infty)$, where $\underline{p}$ is the same as in (2.7). Assume that $N \in \mathbb{N} \cap [N_{X,\,A}, \infty)$ with $N_{X,\,A}$ the same as in (3.2). Further assume that $q \in (\max\{p_0, 1\}, \infty]$ and

$$d \in \left[\left\lfloor \left(\frac{1}{\theta_0} - 1\right) \frac{\ln b}{\ln(\lambda_-)} \right\rfloor, \infty\right) \cap \mathbb{Z}_+. \tag{3.7}$$

(i) An *anisotropic* (X, q, d)*-atom* a is a measurable function on $\mathbb{R}^n$ satisfying that

(i)$_1$ $\operatorname{supp} a := \{x \in \mathbb{R}^n : a(x) \neq 0\} \subset B$, where $B \in \mathcal{B}$ and $\mathcal{B}$ is the same as in (2.4);

(i)$_2$ $\|a\|_{L^q(\mathbb{R}^n)} \le |B|^{\frac{1}{q}} \|\mathbf{1}_B\|_X^{-1}$;

(i)$_3$ $\int_{\mathbb{R}^n} a(x) x^\gamma \, dx = 0$ for any $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| \le d$, here and thereafter, for any $\gamma := \{\gamma_1, \ldots, \gamma_n\} \in \mathbb{Z}^n_+$, $|\gamma| := \gamma_1 + \cdots + \gamma_n$ and $x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n}$.

(ii) The *anisotropic finite atomic Hardy space* $H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$, associated with both A and X, is defined to be the set of all the $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exists a $K \in \mathbb{N}$, a sequence $\{\lambda_i\}_{i=1}^K \subset (0, \infty)$, and a finite sequence $\{a_i\}_{i=1}^K$ of anisotropic (X, q, d)-atoms supported, respectively, in $\{B^{(i)}\}_{i=1}^K \subset \mathcal{B}$ such that

$$f = \sum_{i=1}^K \lambda_i a_i .$$

Moreover, for any $f \in H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$, define

$$\|f\|_{H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)} := \inf \left\| \left\{ \sum_{i=1}^{K} \left[\frac{\lambda_i \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X ,$$

where the infimum is taken over all the decompositions of f as above.

Let A be a dilation and X the same as in Definition 3.10. In the remainder of this article, we always let

$$d_{X,\,A} := \left\lfloor \left(\frac{1}{\theta_0} - 1 \right) \frac{\ln b}{\ln(\lambda_-)} \right\rfloor. \tag{3.8}$$

To establish the dual theorem of $H^A_X(\mathbb{R}^n)$, we need its atomic and its finite atomic characterizations as follows, which are simple corollaries of [84, Theorem 4.3 and Lemma 7.2] and [84, Theorem 5.4], respectively.

Lemma 3.11 *Let A, X, q, d, and θ_0 be the same as in Definition 3.10. Further assume that X has an absolutely continuous quasi-norm, $\{a_j\}_{j\in\mathbb{N}}$ is a sequence of anisotropic (X, q, d)-atoms supported, respectively, in the balls $\{B^{(j)}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ such that*

$$\left\| \left\{ \sum_{j\in\mathbb{N}} \left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right]^{\theta_0} \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{\theta_0}} \right\|_X < \infty.$$

Then the series $f := \sum_{j\in\mathbb{N}} \lambda_j a_j$ converges in $H^A_X(\mathbb{R}^n)$, $f \in H^A_X(\mathbb{R}^n)$, and there exists a positive constant C, independent of f, such that

$$\|f\|_{H^A_X(\mathbb{R}^n)} \le C \left\| \left\{ \sum_{j\in\mathbb{N}} \left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right]^{\theta_0} \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{\theta_0}} \right\|_X .$$

Lemma 3.12 *Let A, X, q, d, θ_0, and p_0 be the same as in Definition 3.10.*

(i) *If $q \in (\max\{p_0, 1\}, \infty)$, then $\|\cdot\|_{H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)}$ and $\|\cdot\|_{H^A_X(\mathbb{R}^n)}$ are equivalent quasi-norms on $H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$;*

(ii) *$\|\cdot\|_{H^{A,\infty,d}_{X,\mathrm{fin}}(\mathbb{R}^n)}$ and $\|\cdot\|_{H^A_X(\mathbb{R}^n)}$ are equivalent quasi-norms on $H^{A,\infty,d}_{X,\mathrm{fin}}(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$, where $\mathcal{C}(\mathbb{R}^n)$ denotes the set of all continuous functions on $\mathbb{R}^n$.*

The following conclusion is also needed for establishing the dual theorem.

Proposition 3.13 *Let A, X, and d be the same as in Definition 3.10. Then the set $H^{A,\infty,d}_{X,\mathrm{fin}}(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$ is dense in $H^A_X(\mathbb{R}^n)$.*

Proof From [84, Lemma 7.2], it easily follows that $H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$ is dense in $H^A_X(\mathbb{R}^n)$. Thus, to show that $H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ is also dense in $H^A_X(\mathbb{R}^n)$, it suffices to prove that the set $H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ is dense in $H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$ with the quasi-norm $\|\cdot\|_{H^A_X(\mathbb{R}^n)}$. To this end, we only need to show that, for any given anisotropic (X,∞,d)-atom a supported in the anisotropic ball $B:=x_0+B_{i_0}$ with $x_0\in\mathbb{R}^n$ and $i_0\in\mathbb{Z}$,

$$\lim_{k\to-\infty}\|a-\varphi_k*a\|_{H^A_X(\mathbb{R}^n)}=0, \tag{3.9}$$

where $\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$ and $\operatorname{supp}\varphi\subset B_0$. Let $s\in(\max\{1,p_0\},\infty)$ with p_0 the same as in Definition 3.1. Observe that, for any $k\in(-\infty,0]\cap\mathbb{Z}$,

$$\frac{|B_{\max\{i_0,0\}+\tau}|^{\frac{1}{s}}(a-\varphi_k*a)}{\|\mathbf{1}_{x_0+B_{\max\{i_0,0\}+\tau}}\|_X\|a-\varphi_k*a\|_{L^s(\mathbb{R}^n)}}$$

is an anisotropic (X,s,d)-atom supported in the anisotropic ball $x_0+B_{\max\{i_0,0\}+\tau}$, which, combined with Lemma 3.11, further implies that

$$\begin{aligned}\|a-\varphi_k*a\|_{H^A_X(\mathbb{R}^n)}&\lesssim\frac{\|\mathbf{1}_{x_0+B_{\max\{i_0,0\}+\tau}}\|_X\|a-\varphi_k*a\|_{L^s(\mathbb{R}^n)}}{|B_{\max\{i_0,0\}+\tau}|^{\frac{1}{s}}}\\&\lesssim\|a-\varphi_k*a\|_{L^s(\mathbb{R}^n)}.\end{aligned}$$

From this and [4, p.15, Lemma 3.8], we infer (3.9), which then completes the proof of Proposition 3.13. □

The following technical lemma is just [4, p. 49, (8.9)] (see also [58, Lemma 3.4]).

Lemma 3.14 *Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $d\in\mathbb{Z}_+$, and B be an anisotropic ball in $\mathfrak{B}$. Then there exists a positive constant C, depending only on d, such that*

$$\sup_{x\in B}\left|P^d_Bf(x)\right|\le C\fint_B|f(x)|\,dx.$$

Now, we prove that the dual space of $H^A_X(\mathbb{R}^n)$ is $\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$.

Theorem 3.15 *Let A, X, q, d, and θ_0 be the same as in Definition 3.10. Further assume that X has an absolutely continuous quasi-norm. Then the dual space of $H^A_X(\mathbb{R}^n)$, denoted by $(H^A_X(\mathbb{R}^n))^*$, is $\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$ with $1/q+1/q'=1$ in the following sense:*

(i) *Let $g\in\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$. Then the linear functional*

$$L_g:\ f\to L_g(f):=\int_{\mathbb{R}^n}f(x)g(x)\,dx, \tag{3.10}$$

initially defined for any $f\in H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$, has a bounded extension to $H^A_X(\mathbb{R}^n)$.

(ii) *Conversely, any continuous linear functional on* $H_X^A(\mathbb{R}^n)$ *arises as in* (3.10) *with a unique* $g \in \mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$.

Moreover, $\|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)} \sim \|L_g\|_{(H_X^A(\mathbb{R}^n))^*}$, *where the positive equivalence constants are independent of* g.

Proof We first show (i) in the case $q \in (\max\{1, p_0\}, \infty)$ with p_0 the same as in Definition 3.10. To this end, let $g \in \mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$. For any $f \in H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$, by Definition 3.10, we know that there exists a sequence $\{\lambda_j\}_{j=1}^m \subset (0,\infty)$ and a sequence $\{a_j\}_{j=1}^m$ of anisotropic $(X,\ q,\ d)$-atoms supported, respectively, in the balls $\{B^{(j)}\}_{j=1}^m \subset \mathcal{B}$ such that $f = \sum_{j=1}^m \lambda_j a_j$ and

$$\left\| \left\{ \sum_{j=1}^m \left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right]^{\theta_0} \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{\theta_0}} \right\|_X \sim \|f\|_{H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)}.$$

From these, the vanishing moments of a_j, the Hölder inequality, the size condition of a_j, Remark 3.4(ii), Lemma 3.12(i), and Definition 3.3 with both q replaced by q' and s replaced by θ_0, it follows that

$$\begin{aligned}
|L_g(f)| &= \left| \int_{\mathbb{R}^n} f(x)g(x)\,dx \right| \le \sum_{j=1}^m \lambda_j \left| \int_{B^{(j)}} a_j(x)g(x)\,dx \right| \\
&= \sum_{j=1}^m \lambda_j \inf_{P\in\mathcal{P}_d(\mathbb{R}^n)} \left| \int_{B^{(j)}} a_j(x)\,[g(x)-P(x)]\,dx \right| \\
&\le \sum_{j=1}^m \lambda_j \|a_j\|_{L^q(\mathbb{R}^n)} \inf_{P\in\mathcal{P}_d(\mathbb{R}^n)} \left[\int_{B^{(j)}} |g(x)-P(x)|^{q'}\,dx \right]^{\frac{1}{q'}} \\
&\le \sum_{j=1}^m \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \inf_{P\in\mathcal{P}_d(\mathbb{R}^n)} \left[\fint_{B^{(j)}} |g(x)-P(x)|^{q'}\,dx \right]^{\frac{1}{q'}} \\
&\lesssim \left\| \left\{ \sum_{i=1}^m \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{\theta_0} \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{\theta_0}} \right\|_X \|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)} \\
&\sim \|f\|_{H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)} \|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)} \sim \|f\|_{H^A_X(\mathbb{R}^n)} \|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)}.
\end{aligned} \tag{3.11}$$

Moreover, by [84, Lemma 7.2] and the assumption that X has an absolutely continuous quasi-norm, we find that $H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$ is dense in $H_X^A(\mathbb{R}^n)$. This, together with (3.11) and a standard density argument, further implies that, when $q \in (\max\{1, p_0\}, \infty)$, (i) holds true and

$$\|L_g\|_{(H_X^A(\mathbb{R}^n))^*} \lesssim \|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)}$$

with the implicit positive constant independent of g.

We next prove (i) in the case $q=\infty$. Indeed, using Proposition 3.13 and repeating the above proof for any given $q\in(\max\{1,p_0\},\infty)$, we then conclude that any $g\in\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ induces a bounded linear functional on $H^A_X(\mathbb{R}^n)$, which is initially defined on $H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ and given by setting, for any $\ell\in H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$,

$$L_g:\ \ell\mapsto\ L_g(\ell):=\int_{\mathbb{R}^n}\ell(x)g(x)\,dx, \tag{3.12}$$

and then has a bounded linear extension to $H^A_X(\mathbb{R}^n)$. Let $g\in\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$. Thus, it remains to show that, for any $f\in H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$,

$$L_g(f)=\int_{\mathbb{R}^n}f(x)g(x)\,dx. \tag{3.13}$$

To this end, suppose $f\in H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$ and $\operatorname{supp}f\subset x_0+B_{i_0}$ with $x_0\in\mathbb{R}^n$ and $i_0\in\mathbb{Z}$. Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp}\varphi\subset B_0$ and $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$. Letting $s\in(\max\{1,p_0\},\infty)$, by the proof of Proposition 3.13, we find that, for any $k\in(-\infty,0]\cap\mathbb{Z}$ and $f\in L^s(\mathbb{R}^n)$,

$$\varphi_k*f\in H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n) \tag{3.14}$$

and

$$\lim_{k\to-\infty}\|f-\varphi_k*f\|_{L^s(\mathbb{R}^n)}=0. \tag{3.15}$$

From this and the Riesz lemma (see, for instance, [30, Theorem 2.30]), it follows that there exists a subsequence $\{k_h\}_{h\in\mathbb{N}}\subset(-\infty,0]\cap\mathbb{Z}$ such that $\lim_{h\to\infty}k_h=-\infty$ and, for almost every $x\in\mathbb{R}^n$,

$$\lim_{h\to\infty}\varphi_{k_h}*f(x)=f(x).$$

By (3.15) and an argument similar to that used in the proof of Proposition 3.13, we conclude that $\lim_{h\to\infty}\|f-\varphi_{k_h}*f\|_{H^A_X(\mathbb{R}^n)}=0$, which, combined with Lemma 3.11, (3.14), (3.12), the fact that

$$\left|\left(\varphi_{k_h}*f\right)g\right|\le\|f\|_{L^\infty(\mathbb{R}^n)}\mathbf{1}_{x_0+B_{\max\{i_0,0\}+\tau}}|g|\in L^1(\mathbb{R}^n),$$

and the Lebesgue dominated convergence theorem (see, for instance, [30, Theorem 2.24]), further implies that

$$\begin{aligned}L_g(f)&=\lim_{h\to\infty}L_g(\varphi_{k_h}*f)=\lim_{h\to\infty}\int_{\mathbb{R}^n}\varphi_{k_h}*f(x)g(x)\,dx\\&=\int_{\mathbb{R}^n}f(x)g(x)\,dx.\end{aligned}$$

This finishes the proof of (3.13) and hence (i) in the case $q = \infty$. Moreover, repeating the proof in (3.11), we obtain, for any $q \in (\max\{1, p_0\}, \infty]$,

$$\|L_g\|_{(H_X^A(\mathbb{R}^n))^*} \lesssim \|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)} \tag{3.16}$$

with the implicit positive constant independent of g.

We next show (ii). For this purpose, let $\pi_B : L^1(B) \to \mathcal{P}_d(\mathbb{R}^n)$, with $B \in \mathcal{B}$, be the natural projection such that, for any $f \in L^1(B)$ and $Q \in \mathcal{P}_d(\mathbb{R}^n)$,

$$\int_B \pi_B(f)(x)Q(x)\,dx = \int_B f(x)Q(x)\,dx. \tag{3.17}$$

For any $q \in (\max\{1, p_0\}, \infty]$ and any ball $B \in \mathcal{B}$, the *closed subspace* $L_0^q(B)$ of $L^q(B)$ is defined by setting

$$L_0^q(B) := \left\{f \in L^q(B) :\ \pi_B(f) = 0 \text{ and } f \neq 0 \text{ almost everywhere on } B\right\},$$

where $L^q(B)$ is the subspace of $L^q(\mathbb{R}^n)$ consisting of all the measurable functions on $\mathbb{R}^n$ vanishing outside B. For any $f \in L_0^q(B)$, since $f \neq 0$ almost everywhere on B, we can easily deduce $\|f\|_{L^q(\mathbb{R}^n)} \neq 0$. Therefore,

$$\frac{|B|^{\frac{1}{q}}}{\|\mathbf{1}_B\|_X}\|f\|^{-1}_{L^q(\mathbb{R}^n)} f$$

is an anisotropic $(X,\ q,\ d)$-atom. From this and Lemma 3.11, it follows that

$$\left\|\frac{|B|^{1/q}}{\|\mathbf{1}_B\|_X}\|f\|^{-1}_{L^q(\mathbb{R}^n)} f\right\|_{H_X^A(\mathbb{R}^n)} \lesssim 1. \tag{3.18}$$

Now, suppose $L \in (H_X^A(\mathbb{R}^n))^*$. Then, by (3.18), we find that, for any $f \in L_0^q(B)$,

$$|L(f)| \le \|L\|_{(H_X^A(\mathbb{R}^n))^*}\frac{\|\mathbf{1}_B\|_X}{|B|^{1/q}}\|f\|_{L^q(\mathbb{R}^n)}. \tag{3.19}$$

Therefore, L provides a bounded linear functional on $L_0^q(B)$. Thus, applying the Hahn–Banach theorem (see, for instance, [30, Theorem 5.6]), we conclude that there exists a linear functional L_B, which extends L to the whole space $L^q(B)$ without increasing its norm.

When $q \in (\max\{1, p_0\}, \infty)$, by the duality $(L^q(B))^* = L^{q'}(B)$, we find that there exists an $h_B \in L^{q'}(B) \subset L^1(B)$ such that, for any $f \in L_0^q(B)$,

$$L(f) = L_B(f) = \int_B f(x)h_B(x)\,dx. \tag{3.20}$$

In the case $q=\infty$, let $\widetilde{q}\in(\max\{1,p_0\},\infty)$. Then there exists an $h_B\in L^{\widetilde{q}'}(B)\subset L^1(B)$ such that, for any $f\in L_0^\infty(B)\subset L^{\widetilde{q}}(B)$, $L(f)=\int_B f(x)h_B(x)\,dx$. Altogether, we conclude that, for any $q\in(\max\{1,p_0\},\infty]$, there exists an $h_B\in L^{q'}(B)$ such that, for any $f\in L_0^q(B)$,

$$L(f)=\int_B f(x)h_B(x)\,dx. \tag{3.21}$$

Next, we prove that such an $h_B\in L^{q'}(B)$ is unique in the sense of modulo $\mathcal{P}_d(\mathbb{R}^n)$. Indeed, assume that $\widetilde{h_B}$ is another function of $L^{q'}(B)$ such that

$$L(f)=\int_B f(x)\widetilde{h_B}(x)\,dx \tag{3.22}$$

for any $f\in L_0^q(B)$. Then, from (3.21), (3.22), and (3.17), we infer that, for any $f\in L^\infty(B)$, $f-\pi_B(f)\in L_0^\infty(B)$ and

$$\begin{aligned}
0&=\int_B [f(x)-\pi_B(f)(x)]\left[h_B(x)-\widetilde{h_B}(x)\right]dx\\
&=\int_B f(x)\left[h_B(x)-\widetilde{h_B}(x)\right]dx-\int_B \pi_B(f)(x)\pi_B(h_B-\widetilde{h_B})(x)\,dx\\
&=\int_B f(x)\left[h_B(x)-\widetilde{h_B}(x)\right]dx-\int_B f(x)\pi_B(h_B-\widetilde{h_B})(x)\,dx\\
&=\int_B f(x)\left[h_B(x)-\widetilde{h_B}(x)-\pi_B(h_B-\widetilde{h_B})(x)\right]dx.
\end{aligned}$$

The arbitrariness of f further implies that $h_B(x)-\widetilde{h_B}(x)=\pi_B(h_B-\widetilde{h_B})(x)$ for almost every $x\in B$. Therefore, after changing the value of h_B (or $\widetilde{h_B}$) on a set of measure zero, we have $h_B-\widetilde{h_B}\in\mathcal{P}_d(\mathbb{R}^n)$. Thus, for any $q\in(\max\{1,p_0\},\infty]$ and $f\in L_0^q(B)$, there exists a unique $h_B\in L^{q'}(B)/\mathcal{P}_d(B)$ such that (3.20) holds true.

For any $j\in\mathbb{R}^n$ and $f\in L_0^q(B_j)$, let g_j be the unique element of $L^{q'}(B_j)/\mathcal{P}_d(B_j)$ such that

$$L(f)=\int_{B_j} f(x)g_j(x)\,dx.$$

Therefore, we can define a local $L^{q'}(\mathbb{R}^n)$ function g by setting $g(x):=g_j(x)$ whenever $x\in B_j$. We point out that, for any $j\in\mathbb{N}$, to obtain the desired g_j, as mentioned above, we may need to change the value of g_j on a set of measure zero. Since we are only dealing with countable balls $\{B_j\}_{j\in\mathbb{N}}$, this does not cause any trouble.

Assume that f is a finite linear combination of anisotropic $(X,\ q,\ d)$-atoms. It is easy to show that there exists an $x_0\in\mathbb{R}^n$ and a $k_0\in\mathbb{Z}$ such that $\operatorname{supp} f\subset x_0+B_{k_0}$. Let

$$j_0:=\frac{\ln A_0+\ln[b^{k_0-1}+\rho(x_0)]}{\ln b}+1.$$

Then, by Definition 2.3, we conclude that $\operatorname{supp} f \subset x_0 + B_{k_0} \subset B_{j_0}$. Thus, $f \in L_0^q(B_{j_0})$ and

$$L(f) = \int_{B_{j_0}} f(x) g_{j_0}(x)\,dx = \int_{\mathbb{R}^n} f(x) g(x)\,dx.$$

From this and (3.19), we deduce that, for any ball $B \in \mathcal{B}$,

$$\|g\|_{(L_0^q(B))^*} \le \frac{\|\mathbf{1}_B\|_X}{|B|^{1/q}} \|L\|_{(H_X^A(\mathbb{R}^n))^*}. \tag{3.23}$$

Moreover, it is known that

$$\|g\|_{(L_0^q(B))^*} = \inf_{P \in \mathcal{P}_d(\mathbb{R}^n)} \|g - P\|_{L^{q'}(B)}$$

(see, for instance, [4, p. 52, (8.12)]), which, combined with Remark 3.7(ii) and (3.23), further implies that

$$\|g\|_{\mathcal{L}^A_{X,q',d}(\mathbb{R}^n)} \sim \sup_{B \in \mathcal{B}} \frac{|B|^{\frac{1}{q}}}{\|\mathbf{1}_B\|_X} \|g\|_{(L_0^q(B))^*} \le \|L\|_{(H_X^A(\mathbb{R}^n))^*}. \tag{3.24}$$

Thus, $g \in \mathcal{L}^A_{X,q',d}(\mathbb{R}^n)$ and, for any finite linear combination f of anisotropic $(X,\ q,\ d)$-atoms,

$$L(f) = \int_{\mathbb{R}^n} f(x) g(x)\,dx.$$

Now, we show that $g \in \mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$ and $\|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)} \lesssim \|L\|_{(H_X^A(\mathbb{R}^n))^*}$. To this end, for any $m \in \mathbb{N}$, $j \in \{1, \ldots, m\}$, $B^{(j)} \in \mathcal{B}$, and $\lambda_j \in (0, \infty)$, let $h_j \in L^q(B^{(j)})$ with $\|h_j\|_{L^q(B^{(j)})} = 1$ be such that

$$\left[\int_{B^{(j)}} \left|g(x) - P^d_{B^{(j)}} g(x)\right|^{q'} dx\right]^{\frac{1}{q'}} = \int_{B^{(j)}} \left[g(x) - P^d_{B^{(j)}} g(x)\right] h_j(x)\,dx \tag{3.25}$$

and, for any $x \in \mathbb{R}^n$, define

$$a_j(x) := \frac{|B^{(j)}|^{\frac{1}{q}} [h_j(x) - P^d_{B^{(j)}} h_j(x)] \mathbf{1}_{B^{(j)}}(x)}{\|\mathbf{1}_{B^{(j)}}\|_X \|h_j - P^d_{B^{(j)}} h_j\|_{L^q(B^{(j)})}}.$$

Then it is easy to find that, for any $j \in \{1, \dots, m\}$, a_j is an anisotropic (X, q, d)-atom. From this and Lemma 3.11, it follows that $\sum_{j=1}^m \lambda_j a_j \in H_X^A(\mathbb{R}^n)$ and

$$\left\| \sum_{j=1}^m \lambda_j a_j \right\|_{H_X^A(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{j=1}^m \left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right]^{\theta_0} \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{\theta_0}} \right\|_X . \tag{3.26}$$

Moreover, by the Minkowski inequality, the assumption that $\|h_j\|_{L^q(B^{(j)})} = 1$, Lemma 3.14, and the Hölder inequality, we find that, for any $j \in \{1, \dots, m\}$,

$$\begin{aligned}
\left\| h_j - P^d_{B^{(j)}} h_j \right\|_{L^q(B^{(j)})} &\le \|h_j\|_{L^q(B^{(j)})} + \left\| P^d_{B^{(j)}} h_j \right\|_{L^q(B^{(j)})} \\
&\lesssim 1 + \left| B^{(j)} \right|^{\frac{1}{q}} \fint_{B^{(j)}} \left| h_j(x) \right| dx \\
&= 1 + \frac{1}{|B^{(j)}|^{\frac{1}{q'}}} \int_{B^{(j)}} \left| h_j(x) \right| dx \\
&\le 1 + \|h_j\|_{L^q(B^{(j)})} \lesssim 1.
\end{aligned}$$

This, together with (3.25), the assumption that $L \in (H_X^A(\mathbb{R}^n))^*$, and (3.26), further implies that

$$\begin{aligned}
&\sum_{j=1}^m \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} \left| g(x) - P^d_{B^{(j)}} g(x) \right|^{q'} dx \right]^{\frac{1}{q'}} \\
&\quad = \sum_{j=1}^m \frac{\lambda_j |B^{(j)}|^{\frac{1}{q}}}{\|\mathbf{1}_{B^{(j)}}\|_X} \int_{B^{(j)}} \left[g(x) - P^d_{B^{(j)}} g(x) \right] h_j(x)\, dx \\
&\quad = \sum_{j=1}^m \frac{\lambda_j |B^{(j)}|^{\frac{1}{q}}}{\|\mathbf{1}_{B^{(j)}}\|_X} \int_{B^{(j)}} \left[h_j(x) - P^d_{B^{(j)}} h_j(x) \right] g(x) \mathbf{1}_{B^{(j)}}(x)\, dx \\
&\quad \lesssim \sum_{j=1}^m \lambda_j \int_{B^{(j)}} a_j(x) g(x)\, dx = \sum_{j=1}^m \lambda_j L(a_j) = L\left(\sum_{j=1}^m \lambda_j a_j \right) \\
&\quad \lesssim \left\| \sum_{j=1}^m \lambda_j a_j \right\|_{H_X^A(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{j=1}^m \left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right]^{\theta_0} \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{\theta_0}} \right\|_X .
\end{aligned}$$

Using this and Definition 3.3, we obtain $g \in \mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$. Moreover, from $g \in \mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$, Proposition 3.9, and (3.24), we infer that

$$\|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)} \sim \|g\|_{\mathcal{L}^A_{X,q',d}(\mathbb{R}^n)} \lesssim \|L\|_{(H_X^A(\mathbb{R}^n))^*}.$$

This finishes the proof of (ii) and hence Theorem 3.15. □

As a consequence of Theorem 3.15, we have the following equivalence of the anisotropic ball Campanato-type function space $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$; we omit the details.

Corollary 3.16 *Let A, X, d, θ_0, and p_0 be the same as in Theorem 3.15 and $q \in [1, \infty)$ when $p_0 \in (0, 1)$, or $q \in [1, p_0')$ when $p_0 \in [1, \infty)$. Then*

$$\mathcal{L}^A_{X,1,d_{X,A},\theta_0}(\mathbb{R}^n) = \mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$$

with equivalent quasi-norms, where $d_{X,A}$ is the same as in (3.8).

Remark 3.17 (i) If $A := 2\, I_{n\times n}$, then Theorem 3.15 and Corollary 3.16 were obtained in [89, Theorem 3.14 and Corollary 3.15], respectively.

(ii) Recently, Yan et al. [85, Theorem 6.6] obtained the dual theorem of the Hardy space $H_Y(\mathcal{X})$ associated with the ball quasi-Banach function space $Y(\mathcal{X})$ on a given space $\mathcal{X}$ of homogeneous type. We point out that, since there exists no linear structure in a general space $\mathcal{X}$ of homogeneous type, one can not introduce the Schwartz function and the polynomial on $\mathcal{X}$. Indeed, any atom in [85] only has zero degree vanishing moment, while the atom in Theorem 3.15 has vanishing moments up to order $d \in [d_{X,\,A}, \infty) \cap \mathbb{N}$ with $d_{X,\,A}$ the same as in (3.8). Thus, although $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type, Theorem 3.15 can not be deduced from [85, Theorem 6.6] and, actually, they can not cover each other.

4 Equivalent characterizations of $\mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$

In this section, applying the dual theorem obtained in Sect. 3, we establish several equivalent characterizations for the anisotropic ball Campanato-type function space $\mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$. This plays an important role in establishing the Carleson measure characterization of $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ in Sect. 6 below.

Theorem 4.1 *Let A, X, q, d, and θ_0 be the same as in Corollary 3.16 and*

$$\varepsilon \in \left(\frac{\ln b}{\ln(\lambda_-)}\left[\frac{2}{s} + d\frac{\ln(\lambda_+)}{\ln b}\right], \infty\right) \tag{4.1}$$

for some $s \in (0, \theta_0)$. Then the following statements are mutually equivalent:

(i) $f \in \mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$;

(ii) $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)} := \sup \left\| \left\{ \sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}} \right\}^{\frac{1}{\theta_0}} \right\|_X^{-1}$$
$$\times \sum_{j=1}^{m} \frac{\lambda_j |x_j + B_{l_j}|}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}$$
$$\times \int_{\mathbb{R}^n} \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}]} + [\rho(x - x_j)]^{1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}}} \, dx$$
$$< \infty, \tag{4.2}$$

where the supremum is taken over all $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$, with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, and $\{\lambda_j\}_{j=1}^m \subset (0, \infty)$.

Moreover, for any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|f\|_{\mathcal{L}^{A}_{X,q,d,\theta_0}(\mathbb{R}^n)} \sim \|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)}$$

with the positive equivalence constants independent of f.

To show Theorem 4.1, we need the following technical lemma, which is a slightly elaborate variant of [82, Lemma 2.13] and which is indeed a simple corollary of the well-known pointwise estimate that $\mathbf{1}_{x_j+B_{k_j+\ell}} \le b^{\ell} \mathcal{M}(\mathbf{1}_{x_j+B_{k_j}})$ for any $\ell \in \mathbb{Z}_+$, any sequence $\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$, and any sequence $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$; we omit the details here.

Lemma 4.2 *Let X be a ball quasi-Banach function space satisfying Assumption 2.10 with $p_- \in (0, \infty)$, $\ell \in \mathbb{Z}_+$, and $s \in (0, \min\{p_-, 1\})$. Then there exists a positive constant C, independent of both ℓ and s, such that, for any sequence $\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ and any sequence $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$,*

$$\left\| \sum_{j\in\mathbb{N}} \mathbf{1}_{x_j+B_{k_j+\ell}} \right\|_X \le C b^{\frac{\ell}{s}} \left\| \sum_{j\in\mathbb{N}} \mathbf{1}_{x_j+B_{k_j}} \right\|_X,$$

where, for any $j \in \mathbb{N}$, B_{k_j} is the same as in (2.3).

Now, we show Theorem 4.1.

Proof of Theorem 4.1 According to Corollary 3.16, to prove the present theorem, we only need to show that, for any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|f\|_{\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)} \sim \|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)}. \tag{4.3}$$

We first prove

$$\|f\|_{\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)}. \tag{4.4}$$

Indeed, by Definition 2.3, we find that, for any $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, $\{\lambda_j\}_{j=1}^m \subset (0, \infty)$, $\varepsilon \in (0, \infty)$, and $j \in \{1, \dots, m\}$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}]} + [\rho(x - x_j)]^{1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}}} \, dx \\
&\quad \geq \int_{x_j+B_{l_j}} \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}]} + [\rho(x - x_j)]^{1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}}} \, dx \\
&\quad \sim \int_{x_j+B_{l_j}} \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}]}} \, dx \\
&\quad = \fint_{x_j+B_{l_j}} \left| f(x) - P^d_{x_j+B_{l_j}} f(x) \right| dx,
\end{aligned}$$

which, together with Definition 3.3 and (4.2), further implies (4.4).

Conversely, from Definitions 2.3 and 3.3, we deduce that, for any $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, $\{\lambda_j\}_{j=1}^m \subset (0, \infty)$,

$$\begin{aligned}
&\sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j}|}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \int_{\mathbb{R}^n} \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}]} + [\rho(x - x_j)]^{1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}}} \, dx \\
&\quad = \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j}|}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \left[\int_{x_j+B_{l_j}} + \sum_{k=0}^{\infty} \int_{x_j+B_{l_j+k+1} \setminus x_j+B_{l_j+k}} \right] \\
&\qquad \times \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}]} + [\rho(x - x_j)]^{1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}}} \, dx \\
&\quad \leq \sum_{j=1}^m \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \int_{x_j+B_{l_j}} \left| f(x) - P^d_{x_j+B_{l_j}} f(x) \right| dx \\
&\qquad + \sum_{j=1}^m \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \sum_{k=0}^{\infty} b^{-k[1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}]} \\
&\qquad \times \int_{x_j+B_{l_j+k+1}} \left| f(x) - P^d_{x_j+B_{l_j}} f(x) \right| dx \\
&\quad \leq \left\| \left\{ \sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}} \right\}^{\frac{1}{\theta_0}} \right\|_X \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} + \mathrm{I},
\end{aligned} \tag{4.5}$$

where

$$\mathrm{I} := \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \sum_{k=0}^{\infty} b^{-k[1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}]} \int_{x_j+B_{l_j+k+1}} \left| f(x) - P^d_{x_j+B_{l_j}} f(x) \right| dx.$$

Obviously, we have

$$\begin{aligned}
\mathrm{I} &\lesssim \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \sum_{k\in\mathbb{N}} b^{-k[1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}]} \int_{x_j+B_{l_j+k}} \left| f(x) - P^d_{x_j+B_{l_j+k}} f(x) \right| dx \\
&\quad + \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \sum_{k\in\mathbb{N}} b^{-k[1+\varepsilon \frac{\ln(\lambda_-)}{\ln b}]} \\
&\quad \times \int_{x_j+B_{l_j+k}} \left| P^d_{x_j+B_{l_j+k}} f(x) - P^d_{x_j+B_{l_j}} f(x) \right| dx. \qquad (4.6)
\end{aligned}$$

Note that, on the one hand, by the definition of minimizing polynomials, (2.6), [45, Lemma 2.19], and Lemma 3.14, we find that, for any $k \in \mathbb{N}$ and $x \in x_j + B_{l_j+k+1}$,

$$\begin{aligned}
&\left| P^d_{x_j+B_{l_j+k}} f(x) - P^d_{x_j+B_{l_j}} f(x) \right| \\
&\quad \le \sum_{\nu=1}^{k} \left| P^d_{x_j+B_{l_j+\nu}} f(x) - P^d_{x_j+B_{l_j+\nu-1}} f(x) \right| \\
&\quad = \sum_{\nu=1}^{k} \left| P^d_{x_j+B_{l_j+\nu-1}} \left(f - P^d_{x_j+B_{l_j+\nu}} f \right)(x) \right| \\
&\quad \le \sum_{\nu=1}^{k} \left\| P^d_{x_j+B_{l_j+\nu-1}} \left(f - P^d_{x_j+B_{l_j+\nu}} f \right) \right\|_{L^\infty(B(x_j, \lambda_+^{l_j+k}))} \\
&\quad \lesssim \sum_{\nu=1}^{k} \left(\frac{\lambda_+^{l_j+k}}{\lambda_-^{l_j+\nu-1}} \right)^d \left\| P^d_{x_j+B_{l_j+\nu-1}} \left(f - P^d_{x_j+B_{l_j+\nu}} f \right) \right\|_{L^\infty(B(x_j, \lambda_-^{l_j+\nu-1}))} \\
&\quad \lesssim \lambda_+^{kd} \sum_{\nu=1}^{k} \frac{1}{|x_j + B_{l_j+\nu-1}|} \int_{x_j+B_{l_j+\nu}} \left| f(y) - P^d_{x_j+B_{l_j+\nu}} f(y) \right| dy; \qquad (4.7)
\end{aligned}$$

on the other hand, from Definition 2.4(ii), the fact that $s \in (0, \theta_0)$, and Lemma 4.2, we infer that, for any $j \in \{1, \ldots, m\}$,

$$\frac{1}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \lesssim b^{\frac{k}{s}} \frac{1}{\|\mathbf{1}_{x_j+B_{l_j+k}}\|_X} \qquad (4.8)$$

and, for any $k \in \mathbb{N}$,

$$\begin{aligned}
&\left\| \left\{ \sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i+k}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i+k}} \right\}^{\frac{1}{\theta_0}} \right\|_X \\
&\quad \le \left\| \left\{ \sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i+k}} \right\}^{\frac{1}{\theta_0}} \right\|_X \\
&\quad \lesssim b^{\frac{k}{s}} \left\| \left\{ \sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}} \right\}^{\frac{1}{\theta_0}} \right\|_X . \qquad (4.9)
\end{aligned}$$

Combining (4.6), (4.7), (4.8), (4.9), Lemma 4.2, $\lambda_-^{\epsilon} = b^{\epsilon \frac{\ln(\lambda_-)}{\ln b}}$, and $\lambda_+^{d} = b^{d \frac{\ln(\lambda_+)}{\ln b}}$, we conclude that

$$\begin{aligned}
&\left\| \left\{ \sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}} \right\}^{\frac{1}{\theta_0}} \right\|_X^{-1} \times \mathrm{I} \\
&\lesssim \sum_{k\in\mathbb{N}} b^{-k[1-\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} \left\| \left\{ \sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i+k}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i+k}} \right\}^{\frac{1}{\theta_0}} \right\|_X^{-1} \\
&\quad \times \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j+k}}\|_X} \int_{x_j+B_{l_j+k}} \left| f(x) - P^d_{x_j+B_{l_j+k}} f(x) \right| dx \\
&\quad + \sum_{k\in\mathbb{N}} \left(\frac{\lambda_+^d}{\lambda_-^{\varepsilon}} b \right)^k \sum_{\nu=1}^{k} b^{\nu(\frac{2}{s}-1)} \left\| \left\{ \sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i+\nu}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i+\nu}} \right\}^{\frac{1}{\theta_0}} \right\|_X^{-1} \\
&\quad \times \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j+\nu}}\|_X} \int_{x_j+B_{l_j+\nu}} \left| f(y) - P^d_{x_j+B_{l_j+\nu}} f(y) \right| dy \\
&\lesssim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \left\{ \sum_{k\in\mathbb{N}} b^{-k[1-\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} + \sum_{k\in\mathbb{N}} \left(\frac{\lambda_+^d}{\lambda_-^{\varepsilon}} b \right)^k \sum_{\nu=1}^{k} b^{\nu(\frac{2}{s}-1)} \right\} \\
&\lesssim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \left\{ \sum_{k\in\mathbb{N}} b^{-k[1-\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} + \sum_{k\in\mathbb{N}} \left(\frac{\lambda_+^d}{\lambda_-^{\varepsilon}} b \right)^k b^{(\frac{2}{s}-1)k} \right\} \\
&\lesssim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \left\{ \sum_{k\in\mathbb{N}} b^{-k[1-\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} + \sum_{k\in\mathbb{N}} b^{-k[\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}-d\frac{\ln(\lambda_+)}{\ln b}]} \right\}
\end{aligned}$$

$$\sim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \sum_{k\in\mathbb{N}} b^{-k[\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}-d\frac{\ln(\lambda_+)}{\ln b}]},$$

which, together with (4.5), (4.2), (4.1), and the arbitrariness of $m \in \mathbb{N}$, $\{x_j+B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, and $\{\lambda_j\}_{j=1}^m \subset (0,\infty)$, further implies that

$$\|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \sum_{k\in\mathbb{N}} b^{-k[\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}-d\frac{\ln(\lambda_+)}{\ln b}]}$$
$$\sim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)}.$$

This, combined with (4.4), proves (4.3) and hence finishes the proof of Theorem 4.1. □

We can obtain one more equivalent characterization of $\mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$ as follows, whose proof is a slight modification of Theorem 4.1; we omit the details.

Theorem 4.3 *If A, X, q, d, θ_0, and ε are the same as in Theorem 4.1, then the conclusion of Theorem 4.1 with m replaced by ∞ still holds true, where the supremum therein is taken over all $\{x_j+B_{l_j}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ with both $\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{l_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$ and over all $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ satisfying*

$$\left\|\left\{\sum_{j\in\mathbb{N}}\left(\frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}\right)^{\theta_0}\mathbf{1}_{x_j+B_{l_j}}\right\}^{\frac{1}{\theta_0}}\right\|_X \in (0,\infty).$$

Remark 4.4 If $A := 2\,I_{n\times n}$, then Theorems 4.1 and 4.3 were obtained in [89, Theorems 4.1 and 4.4], respectively.

5 Littlewood–Paley function characterizations of $H^A_X(\mathbb{R}^n)$

In this section, we establish the characterizations of $H^A_X(\mathbb{R}^n)$ in terms of the anisotropic Lusin area function, the anisotropic Littlewood–Paley g-function, or the anisotropic Littlewood–Paley g^*_λ-function. These are the consequence of the atomic and the finite atomic characterizations of $H^A_X(\mathbb{R}^n)$ obtained in [84] and play important roles in establishing the Carleson measure characterization of $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ in Sect. 6. First, we recall the concepts of both the anisotropic radial maximal function and the anisotropic radial grand maximal function, which were introduced in [4].

Definition 5.1 Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The *anisotropic radial maximal function* $M^0_\varphi(f)$ of f with respect to φ is defined by setting, for any $x\in\mathbb{R}^n$,

$$M^0_\varphi(f)(x) := \sup_{k\in\mathbb{Z}} |f*\varphi_k(x)|.$$

Moreover, for any given $N \in \mathbb{N}$, the *anisotropic radial grand maximal function* $M_N^0(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N^0(f)(x) := \sup_{\varphi \in \mathcal{S}_N(\mathbb{R}^n)} M_\varphi^0(f)(x).$$

In what follows, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\varphi}$ is defined by setting, for any $\xi \in \mathbb{R}^n$,

$$\widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi \imath x \cdot \xi} \, dx,$$

where $\imath := \sqrt{-1}$ and $x \cdot \xi := \sum_{i=1}^n x_i \xi_i$ for any $x := (x_1, \dots, x_n), \xi := (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, $\widehat{f}$ is defined by setting, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle \widehat{f}, \varphi \rangle := \langle f, \widehat{\varphi} \rangle$.

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_k \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \to \infty$ (see, for instance, [31, p. 50]). Let $\mathcal{C}_c^\infty(\mathbb{R}^n)$ denote the collection of all the infinitely differentiable functions with compact support on $\mathbb{R}^n$. The following Calderón reproducing formula is just [7, Proposition 2.14].

Lemma 5.2 *Let $d \in \mathbb{Z}_+$ and A be a dilation. Assume that $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfies*

$$\operatorname{supp} \phi \subset B_0, \int_{\mathbb{R}^n} x^\gamma \phi(x) \, dx = 0 \textit{ for any } \gamma \in \mathbb{Z}_+^n \textit{ with } |\gamma| \le d, \tag{5.1}$$

and there exists a positive constant C such that

$$\left|\widehat{\phi}(\xi)\right| \ge C \textit{ when } \xi \in \left\{x \in \mathbb{R}^n : (2\|A\|)^{-1} \le \rho(x) \le 1\right\}, \tag{5.2}$$

where $\|A\|$ is the same as in (2.2). Then there exists a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

(i) supp $\widehat{\psi}$ *is compact and away from the origin;*
(ii) *for any $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,*

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}\left(\left(A^*\right)^j \xi\right) \widehat{\phi}\left(\left(A^*\right)^j \xi\right) = 1,$$

where A^* denotes the adjoint matrix of A.

Moreover, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, if f vanishes weakly at infinity, then

$$f = \sum_{j \in \mathbb{Z}} f * \psi_j * \phi_j \textit{ in } \mathcal{S}'\left(\mathbb{R}^n\right).$$

The following definitions of the anisotropic Lusin area function, the anisotropic Littlewood–Paley g-function, and the anisotropic Littlewood–Paley g_λ^*-function were introduced in [64, Definition 2.6].

Definition 5.3 Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be the same as in Lemma 5.2. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the *anisotropic Lusin area function* $S(f)$, the *anisotropic Littlewood–Paley g-function* $g(f)$, and the *anisotropic Littlewood–Paley g_λ^*-function* $g_\lambda^*(f)$ with any given $\lambda \in (0, \infty)$ are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$S(f)(x) := \left[\sum_{k\in\mathbb{Z}} b^{-k} \int_{x+B_k} |f * \phi_k(y)|^2 \, dy\right]^{\frac{1}{2}},$$

$$g(f)(x) := \left[\sum_{k\in\mathbb{Z}} |f * \phi_k(x)|^2\right]^{\frac{1}{2}}, \tag{5.3}$$

and

$$g_\lambda^*(f)(x) := \left\{\sum_{k\in\mathbb{Z}} b^{-k} \int_{\mathbb{R}^n} \left[\frac{b^k}{b^k + \rho(x-y)}\right]^{\lambda} |f * \phi_k(y)|^2 \, dy\right\}^{\frac{1}{2}}.$$

We characterize the space $H_X^A(\mathbb{R}^n)$, respectively, in terms of the anisotropic Lusin area function, the anisotropic Littlewood–Paley g-function, and the anisotropic Littlewood–Paley g_λ^*-function as follows.

Theorem 5.4 *Let A be a dilation and X a ball quasi-Banach function space satisfying both Assumption 2.10 with $p_- \in (0, \infty)$ and Assumption 2.12 with the same p_-, $\theta_0 \in (0, \underline{p})$, and $p_0 \in (\theta_0, \infty)$, where $\underline{p}$ is the same as in (2.7). Then $f \in H_X^A(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $\|S(f)\|_X < \infty$. Moreover, for any $f \in H_X^A(\mathbb{R}^n)$,*

$$\|S(f)\|_X \sim \|f\|_{H_X^A(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Theorem 5.5 *Let A and X be the same as in Theorem 5.4. Then $f \in H_X^A(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $\|g(f)\|_X < \infty$. Moreover, for any $f \in H_X^A(\mathbb{R}^n)$,*

$$\|g(f)\|_X \sim \|f\|_{H_X^A(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Moreover, by Theorems 5.4 and 5.5 and an argument similar to that used in the proof of [16, Theorem 4.11], we easily obtain the following result; we omit the details.

Theorem 5.6 *Let A, X, and θ_0 be the same as in Theorem 5.4 and let $\lambda \in (\max\{1, 2/r_+\}, \infty)$, where*

$$r_+ := \sup\{\theta_0 \in (0,\infty) : X \text{ satisfies Assumption 2.12 for this } \theta_0 \text{ and some } p_0 \in (\theta_0, \infty)\}. \tag{5.4}$$

Then $f \in H_X^A(\mathbb{R}^n)$ *if and only if* $f \in \mathcal{S}'(\mathbb{R}^n)$*,* f *vanishes weakly at infinity, and* $\|g_\lambda^*(f)\|_X < \infty$*. Moreover, for any* $f \in H_X^A(\mathbb{R}^n)$*,*

$$\|g_\lambda^*(f)\|_X \sim \|f\|_{H_X^A(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f*,*

To prove Theorem 5.4, we first present the following conclusion which shows that the quasi-norm $\|\cdot\|_X$ of the anisotropic Lusin area functions defined by different ϕ as in Lemma 5.2 are equivalent.

Theorem 5.7 *Let* A *and* X *be the same as in Theorem 5.4 and* $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ *satisfy both* (5.1) *and* (5.2)*. Then, for any* $f \in \mathcal{S}'(\mathbb{R}^n)$ *vanishing weakly at infinity,*

$$\|S_\phi(f)\|_X \sim \|S_\psi(f)\|_X,$$

where $S_\phi(f)$ *is the same as in* (5.3)*,* $S_\psi(f)$ *is the same as in* (5.3) *with* ϕ *replaced by* ψ*, and the positive equivalence constants are independent of* f*.*

To prove Theorem 5.7, we need the following lemma which is just [7, Lemma 2.3] and originates from [18, Theorem 11].

Lemma 5.8 *Let* A *be a dilation. Then there exists a collection*

$$\mathcal{Q} := \left\{Q_\alpha^k \subset \mathbb{R}^n : k \in \mathbb{Z}, \alpha \in I_k\right\}$$

of open subsets, where I_k *is a certain index set, such that*

(i) $|\mathbb{R}^n \setminus \bigcup_\alpha Q_\alpha^k| = 0$ *for each fixed* k *and* $Q_\alpha^k \cap Q_\beta^k = \emptyset$ *for any* $\alpha \neq \beta$*;*
(ii) *for any* α, β, k, ℓ *with* $\ell \geq k$*, either* $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$ *or* $Q_\alpha^\ell \subset Q_\beta^k$*;*
(iii) *for each* (ℓ, β) *and each* $k < \ell$*, there exists a unique* α *such that* $Q_\beta^\ell \subset Q_\alpha^k$*;*
(iv) *there exist a certain negative integer* v *and a certain positive integer* u *such that, for any* Q_α^k *with both* $k \in \mathbb{Z}$ *and* $\alpha \in I_k$*, there exists an* $x_{Q_\alpha^k} \in Q_\alpha^k$ *satisfying that, for any* $x \in Q_\alpha^k$*,*

$$x_{Q_\alpha^k} + B_{vk-u} \subset Q_\alpha^k \subset x + B_{vk+u}.$$

In what follows, for convenience, we call $\mathcal{Q} := \{Q_\alpha^k\}_{k\in\mathbb{Z},\alpha\in I_k}$ in Lemma 5.8 *dyadic cubes* and k the *level*, denoted by $\ell(Q_\alpha^k)$, of the dyadic cube Q_α^k with both $k \in \mathbb{Z}$ and $\alpha \in I_k$.

The following technical lemma is also necessary, which is just [39, Lemma 6.9]. In what follows, for any $\alpha \in \mathbb{R}$, $\lceil\alpha\rceil$ denotes the smallest integer not less than α.

Lemma 5.9 *Let* d *be the same as in* (3.7)*,* v *and* u *the same as in Lemma 5.8(iv), and*

$$\eta \in \left(\frac{\ln b}{\ln b + (d+1)\ln \lambda_-}, 1\right].$$

Then there exists a positive constant C such that, for any $k, i \in \mathbb{Z}$*, any* $\{c_Q\}_{Q\in\mathcal{Q}} \subset [0,\infty)$ *with* $\mathcal{Q}$ *in Lemma 5.8, and any* $x \in \mathbb{R}^n$*,*

$$\sum_{\ell(Q)=\lceil \frac{k-u}{v} \rceil} |Q| \frac{b^{(k\vee i)(d+1)\frac{\ln\lambda_-}{\ln b}}}{[b^{(k\vee i)} + \rho(x - z_Q)]^{(d+1)\frac{\ln\lambda_-}{\ln b}+1}} c_Q$$
$$\leq C b^{-[k-(k\vee i)](\frac{1}{\eta}-1)} \left\{ \mathcal{M} \left[\sum_{\ell(Q)=\lceil \frac{k-u}{v} \rceil} (c_Q)^\eta \mathbf{1}_Q \right](x) \right\}^{\frac{1}{\eta}},$$

where $\ell(Q)$ *denotes the level of* Q*,* $z_Q \in Q$*, and, for any* $k, i \in \mathbb{Z}$*,* $k \vee i := \max\{k, i\}$*.*

We now prove Theorem 5.7.

Proof of Theorem 5.7 By symmetry, to show the present theorem, we only need to prove that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ which vanishes weakly at infinity,

$$\|S_\phi(f)\|_X \lesssim \|S_\psi(f)\|_X. \tag{5.5}$$

To this end, for any $i \in \mathbb{Z}$, $x \in \mathbb{R}^n$, and $y \in x + B_i$, let

$$J_\phi^{(i)}(f)(y) := f * \phi_i(y).$$

Then, by Lemma 5.2 and the Lebesgue dominated convergence theorem, we find that, for any $i \in \mathbb{Z}$, $x \in \mathbb{R}^n$, and $y \in x + B_i$,

$$\begin{aligned} J_\phi^{(i)}(f)(y) &= \sum_{k\in\mathbb{Z}} f * \psi_k * \phi_k * \phi_i(y) \\ &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} f * \psi_k(z) \phi_k * \phi_i(y - z)\, dz \\ &= \sum_{k\in\mathbb{Z}} \sum_{\ell(Q)=\lceil \frac{k-u}{v} \rceil} \int_Q f * \psi_k(z) \phi_k * \phi_i(y - z)\, dz \end{aligned} \tag{5.6}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where all the symbols are the same as in Lemma 5.9. On the other hand, by [8, Lemma 5.4], we conclude that, for any $k, i \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$|\phi_k * \phi_i(x)| \lesssim b^{-(d+1)|k-i|\frac{\ln\lambda_-}{\ln b}} \frac{b^{(k\vee i)(d+1)\frac{\ln\lambda_-}{\ln b}}}{\left[b^{(k\vee i)} + \rho(x)\right]^{(d+1)\frac{\ln\lambda_-}{\ln b}+1}}.$$

This further implies that, for any $Q \in \mathcal{Q}$ with

$$\ell(Q) = \left\lceil \frac{k-u}{v} \right\rceil, \tag{5.7}$$

there exists some $z_Q \in Q$ such that, for any $k, i \in \mathbb{Z}$, $x \in \mathbb{R}^n$, $y \in x + B_i$, and $z \in Q$,

$$|\phi_k * \phi_i(y-z)| \lesssim b^{-(d+1)|k-i|\frac{\ln\lambda_-}{\ln b}} \frac{b^{(k\vee i)(d+1)\frac{\ln\lambda_-}{\ln b}}}{\left[b^{(k\vee i)} + \rho\left(x - z_Q\right)\right]^{(d+1)\frac{\ln\lambda_-}{\ln b}+1}}. \tag{5.8}$$

Moreover, for any $Q \in \mathcal{Q}$ satisfying (5.7), we have $B_{v\ell(Q)+u} \subset B_k$. From this, the Hölder inequality, and Lemma 5.8(iv), we deduce that, for any $z \in Q$,

$$\begin{aligned}\frac{1}{|Q|}\left|\int_Q f * \psi_k(y)\,dy\right| &\le \left[\fint_Q |f * \psi_k(y)|^2\,dy\right]^{\frac12}\\ &\le \left[\frac{1}{|B_{v\ell(Q)-u}|}\int_{z+B_{v\ell(Q)+u}} |f*\psi_k(y)|^2\,dy\right]^{\frac12}\\ &\lesssim \left[b^{-k}\int_{z+B_k} |f*\psi_k(y)|^2\,dy\right]^{\frac12} \sim Y_\psi^{(k)}(f)(z),\end{aligned}$$

where, for any $k \in \mathbb{Z}$ and $z \in \mathbb{R}^n$,

$$Y_\psi^{(k)}(f)(z) := \left[b^{-k}\int_{z+B_k} |f*\psi_k(y)|^2\,dy\right]^{\frac12}.$$

Thus, for any $k \in \mathbb{Z}$ and $Q \in \mathcal{Q}$ satisfying (5.7),

$$\frac{1}{|Q|}\left|\int_Q f*\psi_k(y)\,dy\right| \lesssim \inf_{z\in Q} Y_\psi^{(k)}(f)(z).$$

By this, (5.6), (5.8), and Lemma 5.9, we conclude that, for any given $\eta \in (\frac{\ln b}{\ln b+(d+1)\ln\lambda_-}, 1]$ and for any $i \in \mathbb{Z}$, $x \in \mathbb{R}^n$, and $y \in x + B_i$,

$$\begin{aligned}\left|J_\phi^{(i)}(f)(y)\right| &\lesssim \sum_{k\in\mathbb{Z}} b^{-(d+1)|k-i|\frac{\ln\lambda_-}{\ln b}} \sum_{\ell(Q)=\lceil\frac{k-u}{v}\rceil} |Q|\\ &\quad\times \frac{b^{(k\vee i)(d+1)\frac{\ln\lambda_-}{\ln b}}}{[b^{(k\vee i)} + \rho\left(x-z_Q\right)]^{(d+1)\frac{\ln\lambda_-}{\ln b}+1}} \inf_{z\in Q} Y_\psi^{(k)}(f)(z)\\ &\lesssim \sum_{k\in\mathbb{Z}} b^{-(d+1)|k-i|\frac{\ln\lambda_-}{\ln b}} b^{-[k-(k\vee i)](\frac{1}{\eta}-1)}\\ &\quad\times\left\{\mathcal{M}\left(\sum_{\ell(Q)=\lceil\frac{k-u}{v}\rceil} \inf_{z\in Q}\left[Y_\psi^{(k)}(f)(z)\right]^\eta \mathbf{1}_Q\right)(x)\right\}^{\frac{1}{\eta}}\\ &=: J_{(\eta,i)}(x).\end{aligned} \tag{5.9}$$

Using (3.7), we are able to choose an $\eta \in (\frac{\ln b}{\ln b+(d+1)\ln\lambda_-}, \theta_0)$. Therefore, from (5.9), it follows that, for such an η and any $x \in \mathbb{R}^n$,

$$\left[S_\phi(f)(x)\right]^2 = \sum_{i\in\mathbb{Z}} b^{-i} \int_{x+B_i} \left|J_\phi^{(i)}(f)(y)\right|^2 dy \lesssim \sum_{i\in\mathbb{Z}} \left[J_{(\eta,i)}(x)\right]^2.$$

This, together with the Hölder inequality and the choice that $\eta > \frac{\ln b}{\ln b+(d+1)\ln\lambda_-}$, further implies that, for such an η and any $x \in \mathbb{R}^n$,

$$\begin{aligned}
\left[S_\phi(f)(x)\right]^2 &\lesssim \sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \left\{b^{-(d+1)|k-i|\frac{\ln\lambda_-}{\ln b}} b^{-[k-(k\vee i)](\frac{1}{\eta}-1)}\right\}^2 \\
&\quad \times \sum_{k\in\mathbb{Z}} \left\{\mathcal{M}\left(\sum_{\ell(Q)=\lceil\frac{k-u}{v}\rceil} \inf_{z\in Q}\left[Y_\psi^{(k)}(f)(z)\right]^\eta \mathbf{1}_Q\right)(x)\right\}^{\frac{2}{\eta}} \\
&\lesssim \sum_{k\in\mathbb{Z}} \left\{\mathcal{M}\left(\sum_{\ell(Q)=\lceil\frac{k-u}{v}\rceil} \inf_{z\in Q}\left[Y_\psi^{(k)}(f)(z)\right]^\eta \mathbf{1}_Q\right)(x)\right\}^{\frac{2}{\eta}} \\
&\le \sum_{k\in\mathbb{Z}} \left\{\mathcal{M}\left(\left[Y_\psi^{(k)}(f)\right]^\eta\right)(x)\right\}^{\frac{2}{\eta}}.
\end{aligned}$$

Thus, by the choice that $\eta < \theta_0$ and Assumption 2.10, we find that

$$\begin{aligned}
\left\|S_\phi(f)\right\|_X &\lesssim \left\|\left(\sum_{k\in\mathbb{Z}} \left\{\mathcal{M}\left(\left[Y_\psi^{(k)}(f)\right]^\eta\right)(x)\right\}^{\frac{2}{\eta}}\right)^{\frac{\eta}{2}}\right\|_{X^{\frac{1}{\eta}}}^{\frac{1}{\eta}} \\
&\lesssim \left\|\left(\sum_{k\in\mathbb{Z}} \left[Y_\psi^{(k)}(f)\right]^2\right)^{\frac{1}{2}}\right\|_X = \left\|S_\psi(f)\right\|_X,
\end{aligned}$$

which further implies that (5.5) holds true and hence completes the proof of Theorem 5.7. □

Now, we recall the concept of the anisotropic weight class of Muckenhoupt, associated with a dilation A, which was introduced in [6, Definition 2.4].

Definition 5.10 Let A be a dilation, $p \in [1, \infty)$, and w be a nonnegative measurable function on $\mathbb{R}^n$. The function w is said to belong to the *anisotropic weight class of Muckenhoupt*, $\mathcal{A}_p(A) := \mathcal{A}_p(\mathbb{R}^n, A)$, if there exists a positive constant C such that, when $p \in (1, \infty)$,

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\left\{⨍_{x+B_k} w(y)\,dy\right\}\left\{⨍_{x+B_k} [w(y)]^{-\frac{1}{p-1}}\,dy\right\}^{p-1} \le C$$

or, when $p = 1$,

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\left\{⨍_{x+B_k} w(y)\,dy\right\}\left\{\operatorname*{ess\,sup}_{y\in x+B_k}[w(y)]^{-1}\right\} \le C.$$

Moreover, the minimal constants C as above are denoted by $C_{p,A,n}(w)$.

It is easy to prove that, if $1 \le p \le q \le \infty$, then $\mathcal{A}_p(A) \subset \mathcal{A}_q(A)$. Let

$$\mathcal{A}_\infty(A) := \bigcup_{q\in[1,\infty)} \mathcal{A}_q(A).$$

For any given $w \in \mathcal{A}_\infty(A)$, define the *critical index* q_w of w by setting

$$q_w := \inf\left\{p \in [1,\infty) :\ w \in \mathcal{A}_p(A)\right\}. \tag{5.10}$$

Obviously, $q_w \in [1, \infty)$. By the reverse Hölder inequality (see, for instance, [42, Theorem 1.2]), we conclude that, for any $p \in (1, \infty)$ and $w \in \mathcal{A}_p(A)$, there exists an $\epsilon \in (0, p-1]$ such that $w \in \mathcal{A}_{p-\epsilon}(A)$. Thus, if $q_w \in (1, \infty)$, then $w \notin \mathcal{A}_{q_w}(A)$. Moreover, Johnson and Neugebauer [46, p. 254] gave an example of $w \notin \mathcal{A}_1(A)$ with $A = 2I_{n\times n}$ such that $q_w = 1$.

In what follows, for any nonnegative local integrable function w and any Lebesgue measurable set E, let

$$w(E) := \int_E w(x)\,dx.$$

For any $p \in (0, \infty)$ and any nonnegative local integrable function w, denote by $L^p_w(\mathbb{R}^n)$ the *set of all the measurable functions* f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right\}^{\frac{1}{p}} < \infty.$$

Moreover, let $L^\infty_w(\mathbb{R}^n) := L^\infty(\mathbb{R}^n)$. Obviously, for any $p \in (0, \infty]$ and $w \in \mathcal{A}_\infty(A)$, $L^p_w(\mathbb{R}^n)$ is a ball quasi-Banach function space, which even may not be a quasi-Banach function space (see, for instance, [69, p. 86]).

To show Theorem 5.4, we need the following several technical lemmas. Lemma 5.11 is a direct corollary of [85, Lemma 4.9] (see also [70, (4.6)]) because $(\mathbb{R}^n, \rho, dx)$ is a special space of homogeneous type; Lemma 5.12 is similar to [4, p. 21, Theorem 4.5] and we omit the details of its proof.

Lemma 5.11 *Let A, X, and θ_0 be the same as in Theorem 5.4. Assume that $x_0 \in \mathbb{R}^n$. Then there exists an $\epsilon \in (0,1)$ such that X continuously embeds into $L_w^{\theta_0}(\mathbb{R}^n)$, where $w := [\mathcal{M}(\mathbf{1}_{x_0+B_0})]^\epsilon$ and B_0 is the same as in (2.3) with $k = 0$.*

Lemma 5.12 *Let A and X be the same as in Theorem 5.4. Then $H_X^A(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and the inclusion is continuous.*

Combining Lemmas 5.11 and 5.12, we obtain the following property of $H_X^A(\mathbb{R}^n)$.

Lemma 5.13 *Let A and X be the same as in Theorem 5.4 and let $f \in H_X^A(\mathbb{R}^n)$. Then f vanishes weakly at infinity.*

Proof Let $N \in \mathbb{N}$ be the same as in (3.1). By Lemma 5.12, we find that, for any $k \in \mathbb{Z}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and $y \in x + B_k$, $|f * \varphi_k(x)| \lesssim M_N(f)(y)$. Thus, there exists a positive constant C_1 such that, for any $k \in \mathbb{Z}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,

$$x + B_k \subset \left\{y \in \mathbb{R}^n : \ M_N(f)(y) > C_1|f * \varphi_k(x)|\right\}.$$

By this, Lemma 5.11, and the fact that $w := [\mathcal{M}(\mathbf{1}_{x_0+B_0})]^\epsilon$ with $\epsilon \in (0,1)$ is not integrable on $\mathbb{R}^n$, we conclude that, for any $k \in \mathbb{Z}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,

$$\begin{aligned} |f * \varphi_k(x)| &= [w(B_k)]^{-\frac{1}{\theta_0}} [w(B_k)]^{\frac{1}{\theta_0}} |f * \varphi_k(x)| \\ &\leq [w(B_k)]^{-\frac{1}{\theta_0}} \left[w\left(\left\{y \in \mathbb{R}^n : \ M_N(f)(y) > C_1|f * \varphi_k(x)|\right\}\right)\right]^{\frac{1}{\theta_0}} \\ &\quad \times |f * \varphi_k(x)| \\ &\lesssim [w(B_k)]^{-\frac{1}{\theta_0}} \|M_N(f)\|_{L_w^{\theta_0}(\mathbb{R}^n)} \lesssim [w(B_k)]^{-\frac{1}{\theta_0}} \|M_N(f)\|_X \\ &= [w(B_k)]^{-\frac{1}{\theta_0}} \|f\|_{H_X^A(\mathbb{R}^n)} \to 0 \end{aligned}$$

as $k \to \infty$, which further implies that f vanishes weakly at infinity. This finishes the proof of Lemma 5.13. □

To show Theorem 5.4, we also need the following lemma whose proof is similar to that of [57, Lemma 4.2]; we omit the details here.

Lemma 5.14 *Let A, X, θ_0, and p_0 be the same as in Theorem 5.4, $q \in (\max\{p_0, 1\}, \infty]$, $k_0 \in \mathbb{Z}$, and $\varepsilon \in (0, \infty)$. Assume that $\{\lambda_i\}_{i\in\mathbb{N}} \subset [0,\infty)$, $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathcal{B}$, and $\{m_i^{(\varepsilon)}\}_{i\in\mathbb{N}} \subset L^q(\mathbb{R}^n)$ satisfy that, for any $\varepsilon \in (0,\infty)$ and $i \in \mathbb{N}$,*

$$\operatorname{supp} m_i^{(\varepsilon)} := \left\{x \in \mathbb{R}^n : \ m_i^{(\varepsilon)} \neq 0\right\} \subset A^{k_0} B^{(i)},$$

$$\|m_i^{(\varepsilon)}\|_{L^q(\mathbb{R}^n)} \leq \frac{|B^{(i)}|^{\frac{1}{q}}}{\|\mathbf{1}_{B^{(i)}}\|_X},$$

and

$$\left\|\left\{\sum_{i\in\mathbb{N}} \left[\frac{\lambda_i \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X < \infty.$$

Then

$$\left\| \liminf_{\varepsilon\to 0^+} \left[\sum_{i\in\mathbb{N}} \left|\lambda_i m_i^{(\varepsilon)}\right|^{\theta_0}\right]^{\frac{1}{\theta_0}} \right\|_X \le C \left\| \left\{\sum_{i\in\mathbb{N}} \left[\frac{\lambda_i \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}} \right\|_X,$$

where C is a positive constant independent of λ_i, $B^{(i)}$, $m_i^{(\varepsilon)}$, and ε.

Now, we prove Theorem 5.4.

Proof of Theorem 5.4 Let τ be the same as in (2.5) and u and v the same as in Lemma 5.8(iv). We first show the necessity of the present theorem. To this end, let $f \in H_X^A(\mathbb{R}^n)$. Then, by Lemma 5.13, we find that f vanishes weakly at infinity. On the other hand, it follows from [84, Theorem 4.3] that there exists a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset [0,\infty)$ and a sequence $\{a_i\}_{i\in\mathbb{N}}$ of anisotropic (X, q, d)-atoms supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathcal{B}$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in } \mathcal{S}'\left(\mathbb{R}^n\right)$$

and

$$\|f\|_{H_X^A(\mathbb{R}^n)} \sim \left\| \left\{\sum_{i\in\mathbb{N}} \left[\frac{\lambda_i \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}} \right\|_X.$$

Let a be an (X, q, d)-atom supported in a dyadic cube Q. Let $w := u - v + 2\tau$ and, for any $j \in \mathbb{N}$, $U_j := x_Q + (B_{v[\ell(Q)-j-1]+2\tau} \setminus B_{v[\ell(Q)-j]+2\tau})$. Then, by Lemma 5.8(iv), we conclude that, for any $x \in (A^w Q)^\complement$, there exists some $j_0 \in \mathbb{N}$ such that $x \in U_{j_0}$. For this j_0, choose an $N \in \mathbb{N}$ lager enough such that

$$(N - \beta)vj_0 + \left(\frac{1}{q} - \beta\right)u < 0,$$

where $\beta := (\frac{\ln b}{\ln \lambda_-} + d + 1)\frac{\ln \lambda_-}{\ln b} > \frac{1}{\theta_0}$. By this and an argument similar to that used in the proof of [64, (3.3)], we find that, for any $x \in (A^w Q)^\complement$,

$$S(a)(x) \lesssim b^{Nvj_0} b^{-\frac{v\ell(Q)}{q}} \|a\|_{L^q(Q)}.$$

From this, the size condition of a, and Lemma 5.8(iv), we deduce that, for any $x \in (A^w Q)^\complement$,

$$\begin{aligned} S(a)(x) &\lesssim b^{Nvj_0} b^{-\frac{v\ell(Q)}{q}} \|\mathbf{1}_Q\|_X^{-1} \left|B_{v\ell(Q)+u}\right|^{\frac{1}{q}} \\ &\le b^{(N-\beta)vj_0+(\frac{1}{q}-\beta)u} \|\mathbf{1}_Q\|_X^{-1} \frac{|Q|^\beta}{b^{[\ell(Q)-j_0]v\beta}} \end{aligned}$$

$$\lesssim \|\mathbf{1}_Q\|_X^{-1}\left[\frac{|Q|}{\rho(x-x_Q)}\right]^\beta \le \|\mathbf{1}_Q\|_X^{-1}\left[\mathcal{M}(\mathbf{1}_Q)(x)\right]^\beta.$$

Using this, we obtain, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} S(f)(x) &\le \sum_{i\in\mathbb{N}} |\lambda_i|\, S(a_i)(x)\mathbf{1}_{A^w B^{(i)}}(x) + \sum_{i\in\mathbb{N}} |\lambda_i|\, S(a_i)(x)\mathbf{1}_{(A^w B^{(i)})^\complement}(x) \\ &\lesssim \left\{\sum_{i\in\mathbb{N}} \left[|\lambda_i|\, S(a_i)(x)\mathbf{1}_{A^w B^{(i)}}(x)\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}} \\ &\quad + \sum_{i\in\mathbb{N}} \frac{|\lambda_i|}{\|\mathbf{1}_{B^{(i)}}\|_X}\left[\mathcal{M}\left(\mathbf{1}_{B^{(i)}}\right)(x)\right]^\beta. \end{aligned} \tag{5.11}$$

By (5.11), Assumptions 2.10 and 2.12, and an argument similar to that used in the proof of [84, Theorem 4.3], we further conclude that

$$\|S(f)\|_X \lesssim \|f\|_{H_X^A(\mathbb{R}^n)},$$

which completes the proof of the necessity of the present theorem.

Next, we show the sufficiency of the present theorem. Let ψ and ϕ be the same as in Lemma 5.2 with d in (3.7), f vanish weakly at infinity, and $\|S(f)\|_X < \infty$. Then, from Theorem 5.7, we infer that $S_\psi(f) \in X$. Thus, to show the sufficiency of the present theorem, we need to prove that $f \in H_X^A(\mathbb{R}^n)$ and

$$\|f\|_{H_X^A(\mathbb{R}^n)} \lesssim \left\|S_\psi(f)\right\|_X. \tag{5.12}$$

To this end, for any $k \in \mathbb{Z}$, let $\Omega_k := \{x \in \mathbb{R}^n : S_\psi(f)(x) > 2^k\}$ and

$$\mathcal{Q}_k := \left\{Q \in \mathcal{Q} :\ |Q\cap\Omega_k| > \frac{|Q|}{2} \text{ and } |Q\cap\Omega_{k+1}| \le \frac{|Q|}{2}\right\}.$$

Clearly, for any $Q \in \mathcal{Q}$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \mathcal{Q}_k$. Let $\{Q_i^k\}_i$ be the set of all maximal dyadic cubes in $\mathcal{Q}_k$, that is, there exists no $Q \in \mathcal{Q}_k$ such that $Q_i^k \subsetneqq Q$ for any i. Observe that $\{Q_i^k\}_i$ can be finite and at most countable and hence we omit to indicate the range of i for the simplicity of the below presentation.

For any $Q \in \mathcal{Q}$, let

$$\begin{aligned} \widehat{Q} := \Big\{&(y,t) \in \mathbb{R}_+^{n+1} :\ y \in Q, \\ &b^{v\ell(Q)+u+\tau} \le t < b^{v[\ell(Q)-1]+u+\tau}\Big\}. \end{aligned} \tag{5.13}$$

Obviously, $\{\widehat{Q}\}_{Q\in\mathcal{Q}}$ are mutually disjoint and

$$\mathbb{R}_+^{n+1} = \bigcup_{k\in\mathbb{Z}}\bigcup_i B_{k,i}, \tag{5.14}$$

where, for any $k \in \mathbb{Z}$ and i, $B_{k,i} := \bigcup_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \widehat{Q}$. Then, by Lemma 5.8(ii) and (5.13), we easily find that $\{B_{k,i}\}_{k\in\mathbb{Z},i}$ are also mutually disjoint.

On the other hand, ψ has the vanishing moments up to order d. From Lemma 3.7, the properties of tempered distributions (see, for instance, [34, Theorem 2.3.20]), and (5.14), we deduce that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity and satisfying $\|S(f)\|_X < \infty$ and for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}} f * \psi_k * \phi_k(x) \\ &= \int_{\mathbb{R}_+^{n+1}} (f * \psi_t)(y)\phi_t(x-y)\,dy\,dm(t) \end{aligned} \tag{5.15}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where $m(t)$ denotes the counting measure on $\mathbb{R}$, that is, for any set $E \subset \mathbb{R}$, $m(E)$ is the number of integers contained in E if E has only finitely many elements, or else $m(E) := \infty$. For any $k \in \mathbb{Z}$, i, and $x \in \mathbb{R}^n$, let

$$h_i^k(x) := \int_{B_{k,i}} (f * \psi_t)(y)\phi_t(x-y)\,dy\,dm(t).$$

Next, we prove the sufficiency of the present theorem in three steps.

Step (I) The target of this step is to show that

$$\sum_{k\in\mathbb{Z}} \sum_i h_i^k \text{ converges in } \mathcal{S}'\left(\mathbb{R}^n\right). \tag{5.16}$$

To this end, following the proofs of assertions (i) and (ii) in the proof of the sufficiency of [57, Theorem 3.4(i)] with some slight modifications, we conclude that, for any given $q \in (\max\{p_0, 1\}, \infty)$,

(i) for any $k \in \mathbb{Z}$, i, and $x \in \mathbb{R}^n$,

$$h_i^k(x) = \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \int_{\widehat{Q}} (f * \psi_t)(y)\phi_k(x-y)\,dy\,dm(t)$$

holds true in $L^q(\mathbb{R}^n)$ and hence also in $\mathcal{S}'(\mathbb{R}^n)$;

(ii) for any $k \in \mathbb{Z}$ and i, $h_i^k = \lambda_i^k a_i^k$ is a multiple of an anisotropic (X, q, d)-atom, where, for any $k \in \mathbb{Z}$ and i, $\lambda_i^k \sim 2^k \|\mathbf{1}_{B_i^k}\|_X$ with the positive equivalence constants independent of both k and i, and a_i^k is an anisotropic (X, q, d)-atom satisfying, for any $q \in (\max\{p_0, 1\}, \infty)$, $k \in \mathbb{Z}$, i, and $\gamma \in \mathbb{Z}_+^n$,

$$\begin{aligned} &\operatorname{supp} a_i^k \subset B_i^k := x_{Q_i^k} + B_{v[\ell(Q_i^k)-1]+u+3\tau}, \\ &\|a_i^k\|_{L^q(\mathbb{R}^n)} \le \|\mathbf{1}_{B_i^k}\|_X^{-1} |B_i^k|^{\frac{1}{r}}, \text{ and } \int_{\mathbb{R}^n} a_i^k(x) x^\gamma\,dx = 0. \end{aligned}$$

To show (5.16), we next consider two cases: $i \in \mathbb{N}$ and $i \in \{1, \ldots, I\}$ with some $I \in \mathbb{N}$.

Case 1 $i \in \mathbb{N}$. In this case, to prove (5.16), by Lemma 5.12, it suffices to show that

$$\lim_{l\to\infty} \left\| \sum_{l\le|k|\le m} \sum_{l\le i\le m} \lambda_i^k a_i^k \right\|_{H_X^A(\mathbb{R}^n)} = 0. \tag{5.17}$$

Indeed, for any $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, by the estimate that $\left|Q_i^k \cap \Omega_k\right| \ge \frac{|Q_i^k|}{2}$, we find that, for any $x \in \mathbb{R}^n$,

$$\mathcal{M}\left(\mathbf{1}_{Q_i^k\cap\Omega_k}\right)(x) \gtrsim \fint_{Q_i^k} \mathbf{1}_{Q_i^k\cap\Omega_k}(y)\,dy = \frac{|Q_i^k \cap \Omega_k|}{|Q_i^k|} \ge \frac{1}{2}.$$

This, together with Assumption 2.10, further implies that, for any $l, m \in \mathbb{N}$,

$$\begin{aligned}
\left\| \sum_{l\le|k|\le m} \sum_{l\le i\le m} \left(2^k \mathbf{1}_{B_i^k}\right)^{\theta_0} \right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}} &= \left\| \left[\sum_{l\le|k|\le m} \sum_{l\le i\le m} 2^{k\theta_0} \left(\mathbf{1}_{B_i^k}\right)^2 \right]^{\frac12} \right\|_{X^{\frac{2}{\theta_0}}}^{\frac{2}{\theta_0}} \\
&\lesssim \left\| \left\{ \sum_{l\le|k|\le m} \sum_{l\le i\le m} 2^{k\theta_0} \left[\mathcal{M}\left(\mathbf{1}_{Q_i^k\cap\Omega_k}\right)\right]^2 \right\}^{\frac12} \right\|_{X^{\frac{2}{\theta_0}}}^{\frac{2}{\theta_0}} \\
&\lesssim \left\| \sum_{l\le|k|\le m} \sum_{l\le i\le m} \left(2^k \mathbf{1}_{Q_i^k\cap\Omega_k}\right)^{\theta_0} \right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}}.
\end{aligned} \tag{5.18}$$

In addition, from the fact that, for any $l, m \in \mathbb{N}$, $\sum_{l\le|k|\le m} \sum_{l\le i\le m} \lambda_i^k a_i^k \in H_X^A(\mathbb{R}^n)$, Lemma 3.12(i), and Definition 2.6(i), we deduce that

$$\begin{aligned}
\left\| \sum_{l\le|k|\le m} \sum_{l\le i\le m} \lambda_i^k a_i^k \right\|_{H_X^A(\mathbb{R}^n)} &\lesssim \left\| \left\{ \sum_{l\le|k|\le m} \sum_{l\le i\le m} \left[\frac{\lambda_i^k \mathbf{1}_{B_i^k}}{\|\mathbf{1}_{B_i^k}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X \\
&\sim \left\| \left[\sum_{l\le|k|\le m} \sum_{l\le i\le m} \left(2^k \mathbf{1}_{B_i^k}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X \\
&= \left\| \sum_{l\le|k|\le m} \sum_{l\le i\le m} \left(2^k \mathbf{1}_{B_i^k}\right)^{\theta_0} \right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}}.
\end{aligned} \tag{5.19}$$

On the other hand, it follows from Definition 2.4 that, for any $l, m \in \mathbb{N}$,

$$\begin{aligned}\left\|\left[\sum_{l\le|k|\le m}\left(2^k\mathbf{1}_{\Omega_k}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X^{\theta_0} &= \left\|\left[\sum_{l\le|k|\le m}\left(2^k\mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}+2^k\mathbf{1}_{\Omega_{k+1}}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X^{\theta_0}\\ &\lesssim \left\|\left[\sum_{l\le|k|\le m}\left(2^k\mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X^{\theta_0}\\ &\quad+\left(\frac{1}{2}\right)^{\theta_0}\left\|\left[\sum_{l\le|k|\le m}\left(2^{k+1}\mathbf{1}_{\Omega_{k+1}}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X^{\theta_0}.\end{aligned}$$

Therefore, as $l \to \infty$, we have

$$\left\|\left[\sum_{l\le|k|\le m}\left(2^k\mathbf{1}_{\Omega_k}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X \sim \left\|\left[\sum_{l\le|k|\le m}\left(2^k\mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X. \tag{5.20}$$

This, combined with (5.18) and (5.19), further implies that, as $l \to \infty$,

$$\begin{aligned}&\left\|\sum_{l\le|k|\le m}\sum_{l\le i\le m}\lambda_i^k a_i^k\right\|_{H_X^A(\mathbb{R}^n)}\\ &\quad\lesssim \left\|\sum_{l\le|k|\le m}\sum_{l\le i\le m}\left(2^k\mathbf{1}_{Q_i^k\cap\Omega_k}\right)^{\theta_0}\right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}} \lesssim \left\|\left[\sum_{l\le|k|\le m}\left(2^k\mathbf{1}_{\Omega_k}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X\\ &\quad\sim \left\|\left[\sum_{l\le|k|\le m}\left(2^k\mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X\\ &\quad\le \left\|S_\psi(f)\left(\sum_{l\le|k|\le m}\mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right)^{\frac{1}{\theta_0}}\right\|_X \to 0.\end{aligned}$$

Thus, (5.17) holds true and so (5.16) does in Case 1.

Case 2 $i \in \{1, \dots, I\}$ with some $I \in \mathbb{N}$. In this case, to show (5.16), by Lemma 5.12, it suffices to prove that

$$\lim_{l\to\infty}\left\|\sum_{l\le|k|\le m}\sum_{i=1}^{I}\lambda_i^k a_i^k\right\|_{H_X^A(\mathbb{R}^n)} = 0. \tag{5.21}$$

Indeed, by a proof similar to that of (5.17), it is easy to show that (5.21) also holds true. This finishes the proof of (5.16) in Case 2 and hence (5.16).

Step (II) In this step, we prove that

$$f = \sum_{k\in\mathbb{Z}} \sum_{i} \lambda_i^k a_i^k \text{ in } \mathcal{S}'\left(\mathbb{R}^n\right). \tag{5.22}$$

To this end, for any $x \in \mathbb{R}^n$, let

$$\widetilde{f}(x) := \sum_{k\in\mathbb{Z}} \sum_{i} h_i^k(x) = \sum_{k\in\mathbb{Z}} \sum_{i} \int_{B_{k,i}} (f * \psi_t)(y)\phi_k(x-y)\, dy\, dm(t)$$

in $\mathcal{S}'(\mathbb{R}^n)$, where, for any $k \in \mathbb{Z}$ and i, $B_{k,i}$ is the same as in (5.14). Then, to show (5.22), it suffices to prove that

$$f = \widetilde{f} \text{ in } \mathcal{S}'\left(\mathbb{R}^n\right). \tag{5.23}$$

To this end, by the above assertion (i) and (5.13), we find that, for any given $q \in (\max\{p_0, 1\}, \infty)$ and for any $k \in \mathbb{Z}$, i, and $x \in \mathbb{R}^n$,

$$\begin{aligned} h_i^k(x) &= \lim_{N\to\infty} \int_0^\infty \int_{\mathbb{R}^n} (f * \psi_t)(y)\phi_k(x-y) \\ &\quad \times \mathbf{1}_{\bigcup_{\substack{Q\subset Q_i^k,\, Q\in\mathcal{Q}_k \\ |\ell(Q)|\le N}} \widehat{Q}}(y,t)\, dy\, dm(t) \\ &= \lim_{N\to\infty} \int_{\gamma(N)}^{\eta(N)} \int_{\mathbb{R}^n} (f * \psi_t)(y)\phi_k(x-y)\mathbf{1}_{B_{k,i}}(y,t)\, dy\, dm(t) \end{aligned} \tag{5.24}$$

holds true in $L^q(\mathbb{R}^n)$ and also in $\mathcal{S}'(\mathbb{R}^n)$, where, for any $N \in \mathbb{N}$, $\gamma(N) := b^{vN+u+1}$ and $\eta(N) := b^{-v(N+1)+u+1}$. For the convenience of symbols, we rewrite $\widetilde{f}$ as, for any $x \in \mathbb{R}^n$,

$$\widetilde{f}(x) = \sum_{\ell\in\mathbb{N}} \int_{R^{(\ell)}} (f * \psi_t)(y)\phi_t(x-y)\, dy\, dm(t),$$

where $\{R^{(\ell)}\}_{\ell\in\mathbb{N}}$ is an arbitrary permutation of $\left\{B_{k,i}\right\}_{k\in\mathbb{Z},i}$. For any $L \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let

$$\widetilde{f}_L(x) := f(x) - \sum_{\ell=1}^{L} \int_{R^{(\ell)}} (f * \psi_t)(y)\phi_t(x-y)\, dy\, dm(t).$$

Then, from (5.14), (5.15), and (5.24), it follows that, for any $L \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}\widetilde{f}_L(x) &= \lim_{N\to\infty}\int_{\gamma(N)}^{\eta(N)}\int_{\mathbb{R}^n}(f*\psi_t)(y)\phi_t(x-y)\mathbf{1}_{\cup_{\ell=1}^{\infty}R^{(\ell)}}(y,t)\,dy\,dm(t)\\ &\quad -\lim_{N\to\infty}\int_{\gamma(N)}^{\eta(N)}\int_{\mathbb{R}^n}(f*\psi_t)(y)\phi_t(x-y)\mathbf{1}_{\cup_{\ell=1}^{L}R^{(\ell)}}(y,t)\,dy\,dm(t)\\ &= \lim_{N\to\infty}\int_{\gamma(N)}^{\eta(N)}\int_{\mathbb{R}^n}(f*\psi_t)(y)\phi_t(x-y)\mathbf{1}_{\cup_{\ell=L+1}^{\infty}R^{(\ell)}}(y,t)\,dy\,dm(t)\end{aligned} \tag{5.25}$$

holds true in $\mathcal{S}'(\mathbb{R}^n)$.

Note that $H_X^A(\mathbb{R}^n)$ is continuously embedded into $\mathcal{S}'(\mathbb{R}^n)$ (Lemma 5.12). Thus, to prove (5.23), we only need to show that

$$\|\widetilde{f}_L\|_{H_X^A(\mathbb{R}^n)} \to 0 \text{ as } L\to\infty. \tag{5.26}$$

To do this, we borrow some ideas from the proof of the atomic characterization of $H_X^A(\mathbb{R}^n)$ (see the proof of [84, Theorem 4.3]). Indeed, for any $\varepsilon\in(0,1)$, $L\in\mathbb{N}$, and $x\in\mathbb{R}^n$, let

$$\widetilde{f}_L^{(\varepsilon)}(x) := \int_{\varepsilon}^{\alpha/\varepsilon}\int_{\mathbb{R}^n}(f*\psi_t)(y)\phi_t(x-y)\mathbf{1}_{\cup_{\ell=L+1}^{\infty}R^{(\ell)}}(y,t)\,dy\,dm(t),$$

where $\alpha := b^{-v+2(u+1)}$. Then, by the Lebesgue dominated convergence theorem, we find that, for any $\varepsilon\in(0,1)$, $L\in\mathbb{N}$, and $x\in\mathbb{R}^n$,

$$\begin{aligned}\widetilde{f}_L^{(\varepsilon)}(x) &= \sum_{\ell=L+1}^{\infty}\int_{\varepsilon}^{\alpha/\varepsilon}\int_{\mathbb{R}^n}(f*\psi_t)(y)\phi_t(x-y)\mathbf{1}_{R^{(\ell)}}(y,t)\,dy\,dm(t)\\ &=: \sum_{\ell=L+1}^{\infty}h_\ell^{(\varepsilon)}(x)\end{aligned}$$

in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, by some arguments similar to those used in the proofs of assertions (i) and (ii) in the proof of the sufficiency of [57, Theorem 3.4(i)] with some slight modifications, we conclude that, for any $\varepsilon\in(0,1)$, $q\in(\max\{p_0,1\},\infty)$, $L\in\mathbb{N}$, and $\ell\in\mathbb{N}\cap[L+1,\infty)$, $h_\ell^{(\varepsilon)}$ is a multiple of an anisotropic (X,q,d)-atom, that is, there exists a sequence $\{\lambda_\ell\}_{\ell\in\mathbb{N}\cap(L+1,\infty)}\subset[0,\infty)$ and a sequence $\{a_\ell^{(\varepsilon)}\}_{\ell\in\mathbb{N}\cap(L+1,\infty)}$ of anisotropic (X,q,d)-atoms supported, respectively, in $\{B^{(\ell)}\}_{\ell\in\mathbb{N}\cap(L+1,\infty)}\subset\mathcal{B}$ such that, for any $\ell\in\mathbb{N}\cap[L+1,\infty)$, $h_\ell^{(\varepsilon)}=\lambda_\ell a_\ell^{(\varepsilon)}$, where, for any $\ell\in\mathbb{N}\cap[L+1,\infty)$, λ_ℓ and $B^{(\ell)}$ are independent of ε. Therefore, for any $\varepsilon\in(0,1)$, $L\in\mathbb{N}$, and $x\in\mathbb{R}^n$,

$$\widetilde{f}_L^{(\varepsilon)}(x) = \sum_{\ell=L+1}^{\infty}\lambda_\ell a_\ell^{(\varepsilon)}(x) \text{ in } \mathcal{S}'\left(\mathbb{R}^n\right) \tag{5.27}$$

and

$$\left\| \left\{ \sum_{\ell=L+1}^{\infty} \left[\frac{\lambda_\ell \mathbf{1}_{B^{(\ell)}}}{\|\mathbf{1}_{B^{(\ell)}}\|_X} \right]^{\theta_0} \right\}^{1/\theta_0} \right\|_X < \infty. \tag{5.28}$$

On the other hand, for any given

$$N_0 \in \mathbb{N} \cap \left[\left\lfloor \left(\frac{1}{\theta_0} - 1 \right) \frac{\ln b}{\ln \lambda_-} \right\rfloor + 2, \infty \right),$$

let $M_{N_0}^0$ denote the anisotropic radial grand maximal function in Definition 5.1 with N replaced by N_0. Then, by the just proved conclusion that, for any $\varepsilon \in (0,1)$ and $L \in \mathbb{N}$, $\{a_\ell^{(\varepsilon)}\}_{\ell \in \mathbb{N} \cap (L+1,\infty)}$ is a sequence of anisotropic (X, q, d)-atoms and [84, Lemma 4.7], we find that, for any $\ell \in \mathbb{N} \cap [L+1, \infty)$ and $x \in \mathbb{R}^n$,

$$M_{N_0}^0 \left(a_\ell^{(\varepsilon)} \right)(x) \lesssim M_{N_0}^0 \left(a_\ell^{(\varepsilon)} \right)(x) \mathbf{1}_{A^\tau B^{(\ell)}}(x) + \frac{1}{\|\mathbf{1}_{B^{(\ell)}}\|_X} \left[\mathcal{M} \left(\mathbf{1}_{B^{(\ell)}} \right)(x) \right]^\beta, \tag{5.29}$$

where $\beta := \left(\frac{\ln b}{\ln \lambda_-} + d + 1 \right) \frac{\ln \lambda_-}{\ln b} > \frac{1}{\theta_0}$. Moreover, since $q > 1$, then, from the boundedness of $\mathcal{M}$ on $L^q(\mathbb{R}^n)$ (see [63, Lemma 3.3(ii)]), we deduce that, for any $\varepsilon \in (0,1)$, $L \in \mathbb{N}$, and $\ell \in \mathbb{N} \cap [L+1, \infty)$,

$$\left\| M_{N_0}^0 \left(a_\ell^{(\varepsilon)} \right) \mathbf{1}_{A^\tau B^{(\ell)}} \right\|_{L^q(\mathbb{R}^n)} \lesssim \left\| \mathcal{M} \left(a_\ell^{(\varepsilon)} \right) \mathbf{1}_{A^\tau B^{(\ell)}} \right\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|B^{(\ell)}|^{1/q}}{\|\mathbf{1}_{B^{(\ell)}}\|_X},$$

which, combined with Lemma 5.14, further implies that

$$\begin{aligned} &\left\| \liminf_{\varepsilon \to 0^+} \left\{ \sum_{\ell=L+1}^{\infty} \left[\lambda_\ell M_{N_0}^0 \left(a_\ell^{(\varepsilon)} \right) \mathbf{1}_{A^\tau B^{(\ell)}} \right]^{\theta_0} \right\}^{1/\theta_0} \right\|_X \\ &\quad \lesssim \left\| \left\{ \sum_{\ell=L+1}^{\infty} \left[\frac{\lambda_\ell \mathbf{1}_{B^{(\ell)}}}{\|\mathbf{1}_{B^{(\ell)}}\|_X} \right]^{\theta_0} \right\}^{1/\theta_0} \right\|_X. \end{aligned} \tag{5.30}$$

In addition, let $\varepsilon := \gamma(N)$ with $N \in \mathbb{N} \cap [\lfloor \frac{-u-1}{v} \rfloor + 1, \infty)$. Then, by (5.25), we obtain, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} M_{N_0}^0 \left(\widetilde{f}_L \right)(x) &= M_{N_0}^0 \left(\lim_{N \to \infty} \widetilde{f}_L^{(\gamma(N))} \right)(x) \\ &= \sup_{\varphi \in \mathcal{S}_N(\mathbb{R}^n)} \sup_{k \in \mathbb{Z}} \left| \lim_{N \to \infty} \widetilde{f}_L^{(\gamma(N))} * \varphi_k(x) \right| \\ &\le \liminf_{N \to \infty} \sup_{\varphi \in \mathcal{S}_N(\mathbb{R}^n)} \sup_{k \in \mathbb{Z}} \left| \widetilde{f}_L^{(\gamma(N))} * \varphi_k(x) \right| \end{aligned}$$

$$= \liminf_{N\to\infty} M_{N_0}^0\left(\widetilde{f}_L^{(\gamma(N))}\right).$$

From this, [4, p. 12, Proposition 3.10], (5.27), and (5.29), it follows that, for any $L \in \mathbb{N}$,

$$\begin{aligned}
\|\widetilde{f}_L\|_{H_X^A(\mathbb{R}^n)} &\le \left\|\liminf_{N\to\infty} M_{N_0}^0\left(\widetilde{f}_L^{(\gamma(N))}\right)\right\|_X \\
&\le \left\|\liminf_{N\to\infty} \sum_{\ell=L+1}^{\infty} \lambda_\ell M_{N_0}^0\left(a_\ell^{(\gamma(N))}\right)\right\|_X \\
&\lesssim \left\|\liminf_{N\to\infty} \sum_{\ell=L+1}^{\infty} \lambda_\ell M_{N_0}^0\left(a_\ell^{(\gamma(N))}\right)\mathbf{1}_{A^\tau B^{(\ell)}}\right\|_X \\
&\quad + \left\|\sum_{\ell=L+1}^{\infty} \frac{\lambda_\ell}{\|\mathbf{1}_{A^\tau B^{(\ell)}}\|_X}\left[\mathcal{M}\left(\mathbf{1}_{B^{(\ell)}}\right)\right]^\beta\right\|_X.
\end{aligned}$$

This, together with (5.30), Lemma 3.8, Definition 2.4(ii), Assumption 2.10, and $\beta > \frac{1}{\theta_0}$, further implies that, for any $L \in \mathbb{N}$,

$$\begin{aligned}
\|\widetilde{f}_L\|_{H_X^A(\mathbb{R}^n)} &\lesssim \left\|\liminf_{N\to\infty}\left\{\sum_{\ell=L+1}^{\infty}\left[\lambda_\ell M_{N_0}^0\left(a_\ell^{(\gamma(N))}\right)\mathbf{1}_{A^\tau B^{(\ell)}}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X \\
&\quad + \left\|\left\{\sum_{\ell=L+1}^{\infty} \frac{\lambda_\ell}{\|\mathbf{1}_{B^{(\ell)}}\|_X}\left[\mathcal{M}\left(\mathbf{1}_{B^{(\ell)}}\right)\right]^\beta\right\}^{\frac{1}{\beta}}\right\|_{X^\beta}^{\beta} \\
&\lesssim \left\|\left\{\sum_{\ell=L+1}^{\infty}\left[\frac{\lambda_\ell \mathbf{1}_{B^{(\ell)}}}{\|\mathbf{1}_{B^{(\ell)}}\|_X}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X.
\end{aligned}$$

By this and (5.28), we conclude that (5.26) holds true, which completes the proof of (5.23) and hence (5.22).

Step (III) By (5.22), [84, Theorem 4.3], and some arguments similar to those used in the estimations of both (5.18) and (5.20), we conclude that

$$\begin{aligned}
\|f\|_{H_X^A(\mathbb{R}^n)} &\sim \left\|\left\{\sum_{k\in\mathbb{Z}}\sum_i\left[\frac{\lambda_i^k \mathbf{1}_{B_i^k}}{\|\mathbf{1}_{B_i^k}\|_X}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X = \left\|\left[\sum_{k\in\mathbb{Z}}\sum_i\left(2^k\mathbf{1}_{B_i^k}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X \\
&\lesssim \left\|\sum_{k\in\mathbb{Z}}\sum_i\left(2^k\mathbf{1}_{Q_i^k\cap\Omega_k}\right)^{\theta_0}\right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}} \le \left\|\left[\sum_{k\in\mathbb{Z}}\left(2^k\mathbf{1}_{\Omega_k}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X \\
&\sim \left\|\left[\sum_{k\in\mathbb{Z}}\left(2^k\mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X \le \left\|S_\psi(f)\left[\sum_{k\in\mathbb{Z}}\mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right]^{\frac{1}{\theta_0}}\right\|_X
\end{aligned}$$

$$= \left\| S_\psi(f) \right\|_X ,$$

which further implies that $f \in H_X^A(\mathbb{R}^n)$ and (5.12) holds true. This finishes the proof the sufficiency and hence Theorem 5.4. □

Now, we establish the anisotropic Littlewood–Paley g-function characterization of $H_X^A(\mathbb{R}^n)$. Recall that, for any given dilation A, $\phi \in \mathcal{S}(\mathbb{R}^n)$, $t \in (0, \infty)$, and $j \in \mathbb{Z}$ and for any $f \in \mathcal{S}'(\mathbb{R}^n)$, the *anisotropic Peetre maximal function* $(\phi_j^* f)_t$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$(\phi_j^* f)_t(x) := \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} \frac{|(\phi_{-j} * f)(x+y)|}{[1 + b^j \rho(y)]^t}$$

and the *g-function associated with* $(\phi_j^* f)_t$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$g_{t,*}(f)(x) := \left\{ \sum_{j \in \mathbb{Z}} \left[\left(\phi_j^* f \right)_t (x) \right]^2 \right\}^{1/2} .$$

To prove Theorem 5.5, we need the following estimate which is just [61, Lemma 3.6] originated from [80, (2.66)].

Lemma 5.15 *Let ϕ be the radial function in Lemma 5.2. Then, for any given $N_0 \in \mathbb{N}$ and $\gamma \in (0, \infty)$, there exists a positive constant $C_{(N_0,\gamma)}$, depending only on N_0 and γ, such that, for any $t \in (0, N_0)$, $l \in \mathbb{Z}$, $f \in \mathcal{S}'(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,*

$$\left[\left(\phi_l^* f \right)_t (x) \right]^\gamma \leq C_{(N_0,\gamma)} \sum_{k=0}^{\infty} b^{-kN_0\gamma} b^{k+l} \int_{\mathbb{R}^n} \frac{|\phi_{-(k+l)} * f(y)|^\gamma}{[1 + b^l \rho(x-y)]^{t\gamma}} \, dy.$$

We now prove Theorem 5.5.

Proof of Theorem 5.5 First, let $f \in H_X^A(\mathbb{R}^n)$. Then, by Lemma 5.13, we find that f vanishes weakly at infinity. In addition, repeating the proof of the necessity of Theorem 5.4 with some slight modifications, we easily find that $g(f) \in X$ and $\|g(f)\|_X \lesssim \|f\|_{H_X^A(\mathbb{R}^n)}$. Thus, to prove the present theorem, by Theorem 5.4, we only need to show that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that f vanishes weakly at infinity and $g(f) \in X$,

$$\|S(f)\|_X \lesssim \|g(f)\|_X \tag{5.31}$$

holds true. Notice that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity, any $t \in (0, \infty)$, and almost every $x \in \mathbb{R}^n$, $S(f)(x) \lesssim g_{t,*}(f)(x)$. Thus, to show (5.31), it suffices to prove that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity,

$$\left\| g_{t,*}(f) \right\|_X \lesssim \|g(f)\|_X \tag{5.32}$$

holds true for some $t \in (1/r_+, \infty)$ with r_+ the same as in (5.4). Now, we show (5.32). To this end, assume that $\phi \in \mathcal{S}(\mathbb{R}^n)$ is the radial function in Lemma 5.2. Obviously,

$t \in (1/r_+, \infty)$ implies that there exists a $\theta_0 \in (0, r_+)$ such that $t \in (1/\theta_0, \infty)$. Fix an $N_0 \in (1/\theta_0, \infty)$. By this, Lemma 5.15, and the Minkowski inequality, we find that, for any $x \in \mathbb{R}^n$,

$$
\begin{aligned}
g_{t,*}(f)(x) &= \left\{\sum_{k\in\mathbb{Z}} \left[\left(\phi_k^* f\right)_t(x)\right]^2\right\}^{\frac{1}{2}} \\
&\lesssim \left[\sum_{k\in\mathbb{Z}} \left\{\sum_{j\in\mathbb{Z}_+} b^{-jN_0 r_+} b^{j+k} \int_{\mathbb{R}^n} \frac{|(\phi_{-(j+k)} * f)(y)|^{r_+}}{[1+b^k\rho(x-y)]^{tr_+}}\,dy\right\}^{\frac{2}{r_+}}\right]^{\frac{1}{2}} \\
&\le \left\{\sum_{j\in\mathbb{Z}_+} b^{-j(N_0 r_+ - 1)}\right. \\
&\quad \left.\times \left[\sum_{k\in\mathbb{Z}} b^{\frac{2k}{r_+}} \left\{\int_{\mathbb{R}^n} \frac{|(\phi_{-(j+k)} * f)(y)|^{r_+}}{[1+b^k\rho(x-y)]^{tr_+}}\,dy\right\}^{\frac{2}{r_+}}\right]^{\frac{r_+}{2}}\right\}^{\frac{1}{r_+}},
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
\left\|g_{t,*}(f)\right\|_X^{r_+\theta_0} &\lesssim \left\| \sum_{j\in\mathbb{Z}_+} b^{-j(N_0 r_+ - 1)} \right. \\
&\quad \left. \times \left[\sum_{k\in\mathbb{Z}} b^{\frac{2k}{r_+}} \left\{\int_{\mathbb{R}^n} \frac{|(\phi_{-(j+k)} * f)(y)|^{r_+}}{[1+b^k\rho(\cdot-y)]^{tr_+}}\,dy\right\}^{\frac{2}{r_+}}\right]^{\frac{r_+}{2}} \right\|_{X^{\frac{1}{r_+}}}^{\theta_0} \\
&\le \sum_{j\in\mathbb{Z}_+} b^{-j(N_0 r_+ - 1)\theta_0} \left\| \left\{ \sum_{k\in\mathbb{Z}} b^{\frac{2k}{r_+}} \right.\right. \\
&\quad \times \left[\left(\int_{\{y\in\mathbb{R}^n:\,\rho(\cdot-y)<b^{-k}\}} + \sum_{i\in\mathbb{Z}_+} b^{-itr_+} \int_{\{y\in\mathbb{R}^n:\,b^{i-k-1}<\rho(\cdot-y)<b^{i-k}\}}\right)\right. \\
&\quad \left.\left.\left. \times \left|\left(\phi_{-(j+k)} * f\right)(y)\right|^{r_+}\,dy\right]^{\frac{2}{r_+}} \right\}^{\frac{r_+}{2}} \right\|_{X^{\frac{1}{r_+}}}^{\theta_0} \\
&\le \sum_{j\in\mathbb{Z}_+} b^{-j(N_0 r_+ - 1)\theta_0} \left\| \sum_{k\in\mathbb{Z}} b^{\frac{2k}{r_+}} \left\{ \sum_{i\in\mathbb{N}} b^{-itr_+} \right.\right.
\end{aligned}
$$

$$\times \left[\int_{\{y\in\mathbb{R}^n:\,\rho(\cdot-y)<b^{-k}\}} \left|\left(\phi_{-(j+k)}*f\right)(y)\right|^{r_+}\,dy\right]^{\frac{2}{r_+}} \Bigg\}^{\frac{r_+}{2}} \Bigg\|_{X^{\frac{1}{r_+}}}^{\theta_0}.$$

Then, from the Minkowski inequality again and Assumption 2.10, we further infer that

$$\begin{aligned}
\left\|g_{t,*}(f)\right\|_X^{r_+\theta_0} &\lesssim \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r_+-1)\theta_0} \Bigg\| \sum_{i\in\mathbb{N}} b^{-itr_+} \Bigg\{ \sum_{k\in\mathbb{Z}} b^k \\
&\quad \times \left[\int_{\{y\in\mathbb{R}^n:\,\rho(\cdot-y)<b^{-k}\}} \left|\left(\phi_{-(j+k)}*f\right)(y)\right|^{r_+}\,dy\right]^{\frac{2}{r_+}} \Bigg\}^{\frac{r_+}{2}} \Bigg\|_{X^{\frac{1}{r_+}}}^{\theta_0} \\
&\leq \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r_+-1)\theta_0} \\
&\quad \times \Bigg\| \sum_{i\in\mathbb{N}} b^{(1-tr_+)i} \Bigg\{ \sum_{k\in\mathbb{Z}} \left[\mathcal{M}\left(\left|\phi_{-(j+k)}*f\right|^{r_+}\right)\right]^{\frac{2}{r_+}} \Bigg\}^{\frac{r_+}{2}} \Bigg\|_{X^{\frac{1}{r_+}}}^{\theta_0} \\
&\lesssim \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r_+-1)\theta_0} \sum_{i\in\mathbb{N}} b^{(1-tr_+)i\theta_0} \\
&\quad \times \Bigg\| \Bigg\{ \sum_{k\in\mathbb{Z}} \left[\left|\phi_{-(j+k)}*f\right|^{r_+}\right]^{\frac{2}{r_+}} \Bigg\}^{\frac{r_+}{2}} \Bigg\|_{X^{\frac{1}{r_+}}}^{\theta_0} \\
&\sim \|g(f)\|_X^{r_+\theta_0}.
\end{aligned}$$

This further implies that (5.32) holds true and hence finishes the proof of Theorem 5.5. □

Remark 5.16 (i) If $A := 2\,I_{n\times n}$, then Theorems 5.4, 5.5, and 5.6 were obtained in [16, Theorems 4.9, 4.11, and 4.13] (see also [69, Theorem 3.21] and [82, Theorem 2.10]).

(ii) As was mentioned in Remark 3.17(ii), although $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type, Theorems 5.4, 5.5, and 5.6 can not be deduced from [86, Theorems 4.11, 5.1, and 5.3] and, actually, they can not cover each other.

6 Carleson measure characterization of $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$

In this section, applying the results obtained in previous sections, we establish the Carleson measure characterization of the anisotropic ball Campanato-type function space $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$. To this end, we first introduce the following *anisotropic X-Carleson measure*.

Definition 6.1 Let A be a dilation, X a ball quasi-Banach function space, and $s \in (0, \infty)$. A Borel measure $d\mu$ on $\mathbb{R}^n \times \mathbb{Z}$ is called an *anisotropic X-Carleson measure* if

$$\begin{aligned}\|d\mu\|_X^{A,s} := \sup & \left\| \left\{ \sum_{i=1}^{m} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \\ & \times \sum_{j=1}^{m} \left\{ \frac{\lambda_j |B^{(j)}|^{\frac{1}{2}}}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\int_{\widehat{B^{(j)}}} |d\mu(x,k)| \right]^{\frac{1}{2}} \right\} \\ & < \infty,\end{aligned}$$

where the supremum is taken over all $m \in \mathbb{N}$, $\{B^{(j)}\}_{j=1}^m \subset \mathcal{B}$, and $\{\lambda_j\}_{j=1}^m \subset (0, \infty)$, and, for any $j \in \{1, \ldots, m\}$, $\widehat{B^{(j)}}$ denotes the *tent* over $B^{(j)}$, that is,

$$\widehat{B^{(j)}} := \left\{ (y, k) \in \mathbb{R}^n \times \mathbb{Z} : \; y + B_k \subset B^{(j)} \right\}. \tag{6.1}$$

For the anisotropic X-Carleson measure, we have the following equivalent characterization.

Proposition 6.2 *Let A be a dilation, X a ball quasi-Banach function space, $d\mu$ a Borel measure on $\mathbb{R}^n \times \mathbb{Z}$, $s \in (0, \infty)$, and*

$$\begin{aligned}\|d\mu\|_X^{A,s} := \sup & \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \\ & \times \sum_{j\in\mathbb{N}} \left\{ \frac{\lambda_j |B^{(j)}|^{\frac{1}{2}}}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\int_{\widehat{B^{(j)}}} |d\mu(x,k)| \right]^{\frac{1}{2}} \right\},\end{aligned}$$

where the supremum is taken over all $\{B^{(j)}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ satisfying

$$\left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X \in (0, \infty).$$

Then $\|d\mu\|_X^{A,s} = \|d\mu\|_X^{A,s}$.

Proof Let $d\mu$ be a Borel measure on $\mathbb{R}^n \times \mathbb{Z}$. Obviously, $\|d\mu\|_X^{A,s} \le \|d\mu\|_X^{A,s}$. We next show

$$\|d\mu\|_X^{A,s} \le \|d\mu\|_X^{A,s}. \tag{6.2}$$

Indeed, for any $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}}\subset(0,\infty)$ the same as in the present proposition, by Definition 2.4(iii), we find that

$$\begin{aligned}&\lim_{m\to\infty}\left\|\left\{\sum_{i=1}^{m}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\sum_{j=1}^{m}\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}\\&\quad=\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\sum_{j\in\mathbb{N}}\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}.\end{aligned}$$

Therefore, for any $\varepsilon\in(0,\infty)$, there exists an $m_0\in\mathbb{N}$ such that $\sum_{j=1}^{m_0}\lambda_j\neq0$ and

$$\begin{aligned}&\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\sum_{j\in\mathbb{N}}\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}\\&\quad<\left\|\left\{\sum_{i=1}^{m_0}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\sum_{j=1}^{m_0}\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}+\varepsilon\\&\quad\leq\|d\mu\|_X^{A,s}+\varepsilon.\end{aligned}$$

Combining this, the arbitrariness of both $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}}\subset(0,\infty)$ as in the present proposition, and $\varepsilon\in(0,\infty)$, we further obtain (6.2) and hence complete the proof of Proposition 6.2. □

In what follows, for any given $k\in\mathbb{Z}$, define

$$\delta_k(j):=\begin{cases}1 & \text{when } j=k,\\ 0 & \text{when } j\neq k.\end{cases}$$

Next, we state the main theorem of this section as follows.

Theorem 6.3 *Let A, X, d, and θ_0 be the same as in Theorem 3.15, $p_0\in(\theta_0,2)$, and $\phi\in\mathcal{S}(\mathbb{R}^n)$ be a radial real-valued function satisfying (5.1) and (5.2).*

(i) *If $h\in\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$, then, for any $(x,k)\in\mathbb{R}^n\times\mathbb{Z}$, $d\mu(x,k):=\sum_{\ell\in\mathbb{Z}}|\phi_\ell*h(x)|^2dx\,\delta_\ell(k)$ is an X–Carleson measure on $\mathbb{R}^n\times\mathbb{Z}$; moreover, there exists a positive constant C, independent of h, such that*

$$\|d\mu\|_X^{A,\theta_0}\leq C\|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)}.$$

(ii) *If $h\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and, for any $(x,k)\in\mathbb{R}^n\times\mathbb{Z}$, $d\mu(x,k):=\sum_{\ell\in\mathbb{Z}}|\phi_\ell*h(x)|^2dx\,\delta_\ell(k)$ is an X–Carleson measure on $\mathbb{R}^n\times\mathbb{Z}$, then $h\in\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$*

and, moreover, there exists a positive constant C, independent of h, such that

$$\|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \le C\, \|d\mu\|_X^{A,\theta_0}.$$

Remark 6.4 (i) Note that, if X is a concave ball quasi-Banach function space, then, by Proposition 3.9, Theorem 6.3 gives the Carleson measure characterization of $\mathcal{L}^A_{X,1,d}(\mathbb{R}^n)$.
(ii) If $A := 2\, I_{n\times n}$, then Theorem 6.3 was obtained in [89, Theorem 5.3].

To prove Theorem 6.3, we need the anisotropic tent space associated with ball quasi-Banach function space and its atomic decomposition. We first recall the following concept.

Definition 6.5 Let A be a dilation and, for any $x \in \mathbb{R}^n$, let

$$\Gamma(x) := \{(y,k) \in \mathbb{R}^n \times \mathbb{Z} :\ y \in x + B_k\},$$

which is called the *cone* of aperture 1 with vertex $x \in \mathbb{R}^n$.

Let $\alpha \in (0,\infty)$. For any measurable function $F : \mathbb{R}^n \times \mathbb{Z} \to \mathbb{C}$ and $x \in \mathbb{R}^n$, define

$$\mathscr{A}(F)(x) := \left[\sum_{\ell\in\mathbb{Z}} b^{-\ell} \int_{\{y\in\mathbb{R}^n:\ (y,\ell)\in\Gamma(x)\}} |F(y,\ell)|^2\, dy\right]^{\frac{1}{2}},$$

where $\Gamma(x)$ is the same as in Definition 6.5. A measurable function F on $\mathbb{R}^n \times \mathbb{Z}$ is said to belong to the *anisotropic tent space* $T_2^{A,p}(\mathbb{R}^n \times \mathbb{Z})$, with $p \in (0,\infty)$, if

$$\|F\|_{T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z})} := \|\mathscr{A}(F)\|_{L^p(\mathbb{R}^n)} < \infty.$$

For any given ball quasi-Banach function space X, the *anisotropic X-tent space* $T_X^A(\mathbb{R}^n \times \mathbb{Z})$ is defined to be the set of all the measurable functions F on $\mathbb{R}^n \times \mathbb{Z}$ such that $\mathscr{A}(F) \in X$ and naturally equipped with the quasi-norm $\|F\|_{T_X^A(\mathbb{R}^n\times\mathbb{Z})} := \|\mathscr{A}(F)\|_X$.

We next give the definition of anisotropic (T_X, p)-atoms.

Definition 6.6 Let $p \in (1,\infty)$, A be a dilation, and X a ball quasi-Banach function space. A measurable function $a : \mathbb{R}^n \times \mathbb{Z} \to \mathbb{C}$ is called an *anisotropic* (T_X, p)*-atom* if there exists a ball $B \subset \mathcal{B}$ such that

(i) $\operatorname{supp} a := \{(x,k) \in \mathbb{R}^n \times \mathbb{Z} :\ a(x,k) \neq 0\} \subset \widehat{B}$, where $\widehat{B}$ is the same as in (6.1) with $B^{(j)}$ replaced by B.
(ii) $\|a\|_{T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z})} \le |B|^{1/p}/\|\mathbf{1}_B\|_X$.

Moreover, if a is an anisotropic (T_X, p)-atom for any $p \in (1,\infty)$, then a is called an *anisotropic* (T_X, ∞)*-atom*.

We have the following atomic decomposition on the anisotropic X-tent space $T_X^A(\mathbb{R}^n \times \mathbb{Z})$.

Lemma 6.7 *Let A, X, and θ_0 be the same as in Definition 3.10 and $F:\ \mathbb{R}^n\times\mathbb{Z}\to\mathbb{C}$ a measurable function. If $F\in T_X^A(\mathbb{R}^n\times\mathbb{Z})$, then there exists a sequence $\{\lambda_j\}_{j\in\mathbb{N}}\subset[0,\infty)$, a sequence $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$, and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of anisotropic (T_X,∞)-atoms supported, respectively, in $\{\widehat{B^{(j)}}\}_{j\in\mathbb{N}}$ such that, for almost every $(x,k)\in\mathbb{R}^n\times\mathbb{Z}$,*

$$F(x,k)=\sum_{j\in\mathbb{N}}\lambda_jA_j(x,k),\ |F(x,k)|=\sum_{j\in\mathbb{N}}\lambda_j|A_j(x,k)|$$

pointwisely, and

$$\left\|\left\{\sum_{j\in\mathbb{N}}\left(\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X}\right)^{\theta_0}\mathbf{1}_{B^{(j)}}\right\}^{\frac{1}{\theta_0}}\right\|_X\lesssim\|F\|_{T_X^A(\mathbb{R}^n\times\mathbb{Z})},\tag{6.3}$$

where the implicit positive constant is independent of F.

Proof For any $j\in\mathbb{Z}$, let

$$O_j:=\left\{x\in\mathbb{R}^n:\ \mathscr{A}(F)(x)>2^j\right\},$$

$F_j:=(O_j)^{\complement}$, and, for any given $\gamma\in(0,1)$,

$$(O_j)^*_\gamma:=\left\{x\in\mathbb{R}^n:\mathcal{M}(\mathbf{1}_{O_j})(x)>1-\gamma\right\}.$$

Then, by an argument similar to that used in the proof of [28, (1.14)], we find that

$$\operatorname{supp}F\subset\left[\bigcup_{j\in\mathbb{Z}}\widehat{(O_j)^*_\gamma}\cup E\right],$$

where $E\subset\mathbb{R}^n\times\mathbb{Z}$ satisfies that

$$\sum_{\ell\in\mathbb{Z}}\int_{\{y\in\mathbb{R}^n:\ (y,\ell)\in E\}}dy=0.$$

Moreover, applying [28, (1.15)], we conclude that, for any $j\in\mathbb{Z}$, there exists an integer $N_j\in\mathbb{N}\cup\{\infty\}$, $\{x_k^{(j)}\}_{k=1}^{N_j}\subset(O_j)^*_\gamma$, and $\{l_k\}_{k=1}^{N_j}\subset\mathbb{Z}$ such that $\{x_k^{(j)}+B_{l_k}^{(j)}\}_{k=1}^{N_j}$ has the finite intersection property and

$$\begin{aligned}(O_j)^*_\gamma&=\bigcup_{k=1}^{N_j}\left[x_k^{(j)}+B_{l_k}^{(j)}\right]\\&=\left[x_1^{(j)}+B_{l_1}^{(j)}\right]\cup\left\{\left[x_2^{(j)}+B_{l_2}^{(j)}\right]\setminus\left[x_1^{(j)}+B_{l_1}^{(j)}\right]\right\}\cup\cdots\end{aligned}$$

$$\cup \left\{ \left[x_{N_j}^{(j)} + B_{l_{N_j}}^{(j)} \right] \setminus \bigcup_{i=1}^{N_j-1} \left[x_i^{(j)} + B_{l_i}^{(j)} \right] \right\}$$
$$=: \bigcup_{k=1}^{N_j} B_{j,k}. \tag{6.4}$$

Notice that, for any $j \in \mathbb{Z}$, $\{B_{j,k}\}_{k=1}^{N_j}$ are mutually disjoint. Thus, $\widehat{(O_j)_\gamma^*} = \bigcup_{k=1}^{N_j} \widehat{B_{j,k}}$. For any $j \in \mathbb{Z}$ and $k \in \{1, \dots, N_j\}$, let

$$C_{j,k} := \widehat{B_{j,k}} \cap \left[\widehat{(O_j)_\gamma^*} \setminus \widehat{(O_{j+1})_\gamma^*} \right], \; A_{j,k} := 2^{-j} \left\| \mathbf{1}_{x_k^{(j)} + B_{l_k}^{(j)}} \right\|_X^{-1} F \mathbf{1}_{C_{j,k}}, \tag{6.5}$$

and $\lambda_{j,k} := 2^j \|\mathbf{1}_{x_k^{(j)} + B_{l_k}^{(j)}}\|_X$. Therefore, from (6.4), it follows that

$$F = \sum_{j\in\mathbb{Z}} \sum_{k=1}^{N_j} \lambda_{j,k} A_{j,k} \text{ and } |F| = \sum_{j\in\mathbb{Z}} \sum_{k=1}^{N_j} \lambda_{j,k} \left| A_{j,k} \right|$$

almost everywhere on $\mathbb{R}^n \times \mathbb{Z}$. We now show that, for any $j \in \mathbb{Z}$ and $k \in \{1, \dots, N_j\}$, $A_{j,k}$ is an anisotropic (T_X^A, ∞)-atom supported in $\widehat{x_k^{(j)} + B_{l_k}^{(j)}}$ up to a harmless constant multiple. Obviously,

$$\operatorname{supp} A_{j,k} \subset C_{j,k} \subset \widehat{B_{j,k}} \subset \widehat{x_k^{(j)} + B_{l_k}^{(j)}}.$$

In addition, let $p \in (1, \infty)$ and $h \in T_2^{A,p'}(\mathbb{R}^n \times \mathbb{Z})$ satisfy $\|h\|_{T_2^{A,p'}(\mathbb{R}^n\times\mathbb{Z})} \le 1$. Notice that

$$C_{j,k} \subset \widehat{(O_{j+1})_\gamma^*}^{\complement} = \bigcup_{x\in (O_{j+1})_\gamma^{*\complement}} \Gamma(x).$$

Applying this, [28, Lemma 1.3], the Hölder inequality, and (6.5), we find that

$$\begin{aligned}
\left|\langle A_{j,k}, h\rangle\right| &= \left| \sum_{\ell\in\mathbb{Z}} \int_{\mathbb{R}^n} A_{j,k}(y,\ell) h(y,\ell) \mathbf{1}_{C_{j,k}}(y,\ell)\, dy \right| \\
&\le \sum_{\ell\in\mathbb{Z}} \int_{(y,\ell)\in \bigcup_{x\in(O_{j+1})_\gamma^{*\complement}} \Gamma(x)} \left| A_{j,k}(y,\ell) h(y,\ell) \right| \, dy\, \delta_i(\ell) \\
&\lesssim \int_{(O_{j+1})^\complement} \left[\sum_{\ell\in\mathbb{Z}} \int_{\{y\in\mathbb{R}^n:\ (y,\ell)\in\Gamma(x)\}} b^{-\ell} \left| A_{j,k}(y,\ell) h(y,\ell) \right| \, dy \right] dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{(O_{j+1})^{\complement}} \mathscr{A}\left(A_{j,k}\right)(x)\mathscr{A}(h)(x)\,dx \\
&\le \left\{\int_{(O_{j+1})^{\complement}} \left[\mathscr{A}\left(A_{j,k}\right)(x)\right]^p\,dx\right\}^{\frac{1}{p}} \left\{\int_{(O_{j+1})^{\complement}} [\mathscr{A}(h)(x)]^{p'}\,dx\right\}^{\frac{1}{p'}} \\
&\le 2^{-j}\left\|\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}\right\|_X^{-1} \left\{\int_{(x_k^{(j)}+B_{l_k}^{(j)})\cap(O_{j+1})^{\complement}} [\mathscr{A}(F)(x)]^p\,dx\right\}^{\frac{1}{p}} \\
&\quad\times \|h\|_{T_2^{A,p'}(\mathbb{R}^n\times\mathbb{Z})} \\
&\lesssim \frac{|x_k^{(j)}+B_{l_k}^{(j)}|^{\frac{1}{p}}}{\|\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}\|_X},
\end{aligned}
$$

which, combined with $(T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z}))^* = T_2^{A,p'}(\mathbb{R}^n\times\mathbb{Z})$ (see [22, Theorem 2]), further implies that

$$
\|A_{j,k}\|_{T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z})} \lesssim \frac{|x_k^{(j)}+B_{l_k}^{(j)}|^{\frac{1}{p}}}{\|\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}\|_X}.
$$

Using this, we find that, for any $j\in\mathbb{Z}$ and $k\in\{1,\dots,N_j\}$, $A_{j,k}$ is an anisotropic (T_X^A, p)-atom up to a harmless constant multiple for any $p\in(1,\infty)$. Thus, for any $j\in\mathbb{Z}$ and $k\in\{1,\dots,N_j\}$, $A_{j,k}$ is an anisotropic (T_X^A,∞)-atom up to a harmless constant multiple.

We next prove (6.3). To achieve this, from (6.4), the finite intersection property of $\{x_k^{(j)}+B_{l_k}^{(j)}\}_{k=1}^{N_j}$, the estimate that $\mathbf{1}_{(O_j)_\gamma^*}\lesssim[\mathcal{M}(\mathbf{1}_{O_j})]^{\frac{1}{\theta_0}}$, and Assumption 2.10, we deduce that

$$
\begin{aligned}
&\left\|\left\{\sum_{j\in\mathbb{Z}}\sum_{k=1}^{N_j}\left[\frac{\lambda_{j,k}}{\|\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}\|_X}\right]^{\theta_0}\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}\right\}^{\frac{1}{\theta_0}}\right\|_X \\
&\quad= \left\|\left\{\sum_{j\in\mathbb{Z}}\sum_{k=1}^{N_j}\left[2^j\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X \\
&\quad\lesssim \left\|\left\{\sum_{j\in\mathbb{Z}}\left[2^j\mathbf{1}_{(O_j)_\gamma^*}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X \lesssim \left\|\sum_{j\in\mathbb{Z}}\left\{2^j\left[\mathcal{M}\left(\mathbf{1}_{O_j}\right)\right]^{\frac{1}{\theta_0}}\right\}^{\theta_0}\right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}} \\
&\quad\lesssim \left\|\left\{\sum_{j\in\mathbb{Z}}\left(2^j\mathbf{1}_{O_j}\right)^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X \sim \left\|\left\{\sum_{j\in\mathbb{Z}}\left(2^j\mathbf{1}_{O_j\setminus O_{j+1}}\right)^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X
\end{aligned}
$$

$$\leq \left\| \mathscr{A}(F) \left[\sum_{j\in\mathbb{Z}} \mathbf{1}_{O_j \setminus O_{j+1}} \right]^{\frac{1}{\theta_0}} \right\|_X = \|\mathscr{A}(F)\|_X = \|F\|_{T_X^A(\mathbb{R}^n\times\mathbb{Z})}.$$

This further implies that (6.3) holds true and hence finishes the proof of Lemma 6.7. □

We now prove Theorem 6.3.

Proof of Theorem 6.3 We first show (i). To this end, let $h \in \mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ and $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with $m \in \mathbb{N}$, $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$, and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$. Then we easily find that, for any $j \in \{1, \ldots, m\}$,

$$\begin{aligned} h &= P^d_{x_j+B_{l_j}} h + \left(h - P^d_{x_j+B_{l_j}} h\right) \mathbf{1}_{x_j+B_{l_j+\tau}} + \left(h - P^d_{x_j+B_{l_j}} h\right) \mathbf{1}_{(x_j+B_{l_j+\tau})^\complement} \\ &=: h_j^{(1)} + h_j^{(2)} + h_j^{(3)}, \end{aligned} \tag{6.6}$$

where τ is the same as in (2.5). Let $j \in \{1, \ldots, m\}$. For $h_j^{(1)}$, by the fact that $\int_{\mathbb{R}^n} \phi(x) x^\alpha \, dx = 0$ for any $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq d$, we conclude that, for any $k \in \mathbb{Z}$, $\phi_k * h_j^{(1)} \equiv 0$ and hence

$$\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} \left|\phi_k * h_j^{(1)}(x)\right|^2 dx = 0. \tag{6.7}$$

For $h_j^{(2)}$, from the Tonelli theorem and the boundedness on $L^2(\mathbb{R}^n)$ of the following anisotropic g-function

$$g(h_j^{(2)}) := \left[\sum_{k\in\mathbb{Z}} \left|\phi_k * h_j^{(2)}\right|^2\right]^{\frac{1}{2}}$$

(see, for instance, [39, Theorem 6.3]), we infer that

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} \left|\phi_k * h_j^{(2)}(x)\right|^2 dx \\ &\quad\leq \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left|\phi_k * h_j^{(2)}(x)\right|^2 dx \lesssim \left\|h_j^{(2)}\right\|^2_{L^2(\mathbb{R}^n)} \\ &\quad= \int_{x_j+B_{l_j+\tau}} \left|h(x) - P^d_{x_j+B_{l_j}} h(x)\right|^2 dx \\ &\quad\leq \int_{x_j+B_{l_j+\tau}} \left|h(x) - P^d_{x_j+B_{l_j+\tau}} h(x)\right|^2 dx \\ &\qquad+ \int_{x_j+B_{l_j+\tau}} \left|P^d_{x_j+B_{l_j+\tau}} h(x) - P^d_{x_j+B_{l_j}} h(x)\right|^2 dx. \end{aligned} \tag{6.8}$$

In addition, using Lemma 3.14, we obtain, for any $x \in x_j + B_{l_{j+\tau}}$,

$$\begin{aligned}&\left|P^d_{x_j+B_{l_{j+\tau}}} h(x) - P^d_{x_j+B_{l_j}} h(x)\right| \\ &\quad = \left|P^d_{x_j+B_{l_j+\tau}} \left(h - P^d_{x_j+B_{l_j}} h\right)(x)\right| \\ &\quad \lesssim \frac{1}{|x_j + B_{l_j}|} \int_{x_j+B_{l_{j+\tau}}} \left|h(y) - P^d_{x_j+B_{l_{j+\tau}}} h(y)\right| \, dy.\end{aligned}$$

Thus, combining this with (6.8), Lemma 4.2, and Definition 2.4(ii), we find that, for any $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, and $\{\lambda_j\}_{j=1}^m \subset (0, \infty)$,

$$\begin{aligned}&\left\|\left\{\sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1} \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j}|^{\frac{1}{2}}}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \\ &\quad \times \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} \left|\phi_k * h_j^{(2)}(x)\right|^2 dx\right]^{\frac{1}{2}} \\ &\quad \lesssim J_1 \left\|\left\{\sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_{i+\tau}}}\|_X}\right)^{\theta_0} \mathbf{1}_{x_i+B_{l_{i+\tau}}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1} \\ &\quad \times \sum_{j=1}^m J_2^{(j)} \left\{\left[\int_{x_j+B_{l_{j+\tau}}} \left|h(x) - P^d_{x_j+B_{l_{j+\tau}}} h(x)\right|^2 dx\right]^{\frac{1}{2}}\right. \\ &\quad \left. + \frac{1}{|x_j + B_{l_j}|^{\frac{1}{2}}} \int_{x_j+B_{l_{j+\tau}}} \left|h(x) - P^d_{x_j+B_{l_{j+\tau}}} h(x)\right| dx\right\} \\ &\quad \lesssim \left\|\left\{\sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_{i+\tau}}}\|_X}\right)^{\theta_0} \mathbf{1}_{x_i+B_{l_{i+\tau}}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1} \\ &\quad \times \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_{j+\tau}}|^{\frac{1}{2}}}{\|\mathbf{1}_{x_j+B_{l_{j+\tau}}}\|_X} \left\{\left[\int_{x_j+B_{l_{j+\tau}}} \left|h(x) - P^d_{x_j+B_{l_{j+\tau}}} h(x)\right|^2 dx\right]^{\frac{1}{2}}\right. \\ &\quad \left. + \frac{1}{|x_j + B_{l_j}|^{\frac{1}{2}}} \int_{x_j+B_{l_{j+\tau}}} \left|h(x) - P^d_{x_j+B_{l_{j+\tau}}} h(x)\right| dx\right\} \\ &\quad \le \|h\|_{\mathcal{L}^A_{X,2,d,\theta_0}(\mathbb{R}^n)} + \|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)},\end{aligned}$$

where

$$J_1 := \frac{\|\{\sum_{i=1}^m (\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i+\tau}}\|_X})^{\theta_0} \mathbf{1}_{x_i+B_{l_i+\tau}}\}^{\frac{1}{\theta_0}}\|_X}{\|\{\sum_{i=1}^m (\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X})^{\theta_0} \mathbf{1}_{x_i+B_{l_i}}\}^{\frac{1}{\theta_0}}\|_X}$$

and, for any $j \in \{1, \ldots, m\}$,

$$J_2^{(j)} := \frac{\|\mathbf{1}_{x_j+B_{l_j+\tau}}\|_X}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \frac{\lambda_j |x_j + B_{l_j+\tau}|^{\frac{1}{2}}}{\|\mathbf{1}_{x_j+B_{l_j+\tau}}\|_X}.$$

This, combined with $p_0 \in (\theta_0, 2)$ and Corollary 3.16, further implies that

$$\begin{aligned}
&\left\| \left\{ \sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}} \right\}^{\frac{1}{\theta_0}} \right\|_X^{-1} \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j}|^{\frac{1}{2}}}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \\
&\quad \times \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} \left| \phi_k * h_j^{(2)}(x) \right|^2 dx \right]^{\frac{1}{2}} \\
&\quad \lesssim \|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)}.
\end{aligned} \tag{6.9}$$

Finally, we deal with $h_j^{(3)}$. To do this, letting $s \in (0, \theta_0)$ and $\varepsilon \in (\frac{\ln b}{\ln(\lambda_-)}[\frac{2}{s} + d\frac{\ln(\lambda_+)}{\ln b}], \infty)$, we have, for any $j \in \{1, \ldots, m\}$ and $(x, k) \in \widehat{x_j + B_{l_j}}$,

$$\begin{aligned}
\left|\phi_k * h_j^{(3)}(x)\right| &\lesssim \int_{(x_j+B_{l_j+\tau})^\complement} \frac{b^{\varepsilon k \frac{\ln\lambda_-}{\ln b}}}{[b^k + \rho(x-y)]^{1+\varepsilon\frac{\ln\lambda_-}{\ln b}}} \left|h(y) - P^d_{x_j+B_{l_j}} h(y)\right| dy \\
&\sim \int_{(x_j+B_{l_j+\tau})^\complement} \frac{b^{\varepsilon k \frac{\ln\lambda_-}{\ln b}}}{[b^k + \rho(x_j-y)]^{1+\varepsilon\frac{\ln\lambda_-}{\ln b}}} \left|h(y) - P^d_{x_j+B_{l_j}} h(y)\right| dy \\
&\le \frac{b^{\varepsilon k \frac{\ln\lambda_-}{\ln b}}}{b^{\varepsilon l_j \frac{\ln\lambda_-}{\ln b}}} \int_{(x_j+B_{l_j+\tau})^\complement} \frac{b^{\varepsilon l_j \frac{\ln\lambda_-}{\ln b}}}{[\rho(x_j-y)]^{1+\varepsilon\frac{\ln\lambda_-}{\ln b}}} \left|h(y) - P^d_{x_j+B_{l_j}} h(y)\right| dy \\
&\lesssim \frac{b^{\varepsilon k \frac{\ln\lambda_-}{\ln b}}}{b^{\varepsilon l_j \frac{\ln\lambda_-}{\ln b}}} \int_{(x_j+B_{l_j+\tau})^\complement} \frac{b^{\varepsilon l_j \frac{\ln\lambda_-}{\ln b}}}{b^{l_j(1+\varepsilon\frac{\ln\lambda_-}{\ln b})} + [\rho(x_j-y)]^{1+\varepsilon\frac{\ln\lambda_-}{\ln b}}} \\
&\quad \times \left|h(y) - P^d_{x_j+B_{l_j}} h(y)\right| dy.
\end{aligned}$$

From this and Theorem 4.1, it follows that, for any $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, and $\{\lambda_j\}_{j=1}^m \subset (0, \infty)$,

$$\begin{aligned}
&\left\| \left\{ \sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}} \right\}^{\frac{1}{\theta_0}} \right\|_X^{-1} \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j}|^{\frac{1}{2}}}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \\
&\quad \times \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:(x,k)\in\widehat{x_j+B_{l_j}}\}} \left|\phi_k * h_j^{(3)}(x)\right|^2 \, dx \right]^{\frac{1}{2}} \\
&\lesssim \left\| \left\{ \sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}} \right\}^{\frac{1}{\theta_0}} \right\|_X^{-1} \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j}|}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \\
&\quad \times \sum_{k=-\infty}^{l_j} b^{-(l_j-k)\varepsilon\frac{\ln\lambda_-}{\ln b}} \int_{(x_j+B_{l_j+\tau})^\complement} \frac{b^{\varepsilon l_j \frac{\ln\lambda_-}{\ln b}} |h(x) - P^d_{x_j+B_{l_j}} h(x)|}{b^{l_j(1+\varepsilon\frac{\ln\lambda_-}{\ln b})} + [\rho(x_j - x)]^{1+\varepsilon\frac{\ln\lambda_-}{\ln b}}} \, dx \\
&\lesssim \|h\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)} \sim \|h\|_{\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)}.
\end{aligned}$$

Combining this, (6.6), (6.7), and (6.9), we conclude that

$$\begin{aligned}
&\left\| \left\{ \sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}} \right\}^{\frac{1}{\theta_0}} \right\|_X^{-1} \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j}|^{\frac{1}{2}}}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \\
&\quad \times \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} |\phi_k * h(x)|^2 \, dx \right]^{\frac{1}{2}} \\
&\lesssim \|h\|_{\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)},
\end{aligned}$$

which, together with the arbitrariness of $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, and $\{\lambda_j\}_{j=1}^m \subset (0, \infty)$, further implies that, for any $(x, k) \in \mathbb{R}^n \times \mathbb{Z}$,

$$d\mu(x, k) := |\phi_k * h(x)|^2 \, dx$$

is an X–Carleson measure on $\mathbb{R}^n \times \mathbb{Z}$. Moreover, there exists a positive constant C, independent of b, such that $\|d\mu\|_X^{A,\theta_0} \lesssim \|h\|_{\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)}$. This finishes the proof of (i).

We now prove (ii). To this end, let $f \in H^{A,\infty,d,\theta_0}_{X,\mathrm{fin}}(\mathbb{R}^n)$ with the quasi-norm greater than zero. Then $f \in L^\infty(\mathbb{R}^n)$ with compact support. From this, the assumption that

$h \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, and [28, (2.10)], it follows that

$$\left|\int_{\mathbb{R}^n} f(x)\overline{h(x)}\,dx\right| \sim \left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n} \phi_k * f(x)\overline{\phi_k * h(x)}\,dx\right|. \tag{6.10}$$

In addition, by the assumption that $f \in H^A_X(\mathbb{R}^n)$ and Theorem 5.4, we find that

$$\|\phi_k * f\|_{T^A_X(\mathbb{R}^n\times\mathbb{Z})} \sim \|f\|_{H^A_X(\mathbb{R}^n)} < \infty,$$

which, combined with Lemma 6.7, further implies that there exists a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of anisotropic (T^A_X,∞)-atoms supported, respectively, in $\{\widehat{x_j + B_{l_j}}\}_{j\in\mathbb{N}}$ with $\{x_j + B_{l_j}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ such that, for almost every $(x,k) \in \mathbb{R}^n \times \mathbb{Z}$,

$$\phi_k * f(x) = \sum_{j\in\mathbb{N}} \lambda_j A_j(x,k)$$

and

$$0 < \left\|\left\{\sum_{j\in\mathbb{N}}\left(\frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}\right)^{\theta_0}\mathbf{1}_{x_j+B_{l_j}}\right\}^{\frac{1}{\theta_0}}\right\|_X \lesssim \|f\|_{H^A_X(\mathbb{R}^n)}.$$

From this, (6.10), the Hölder inequality, the size condition of A_j, and the Tonelli theorem, we infer that, for any $f \in H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)\overline{h(x)}\,dx\right| \\
&\quad\lesssim \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\lambda_j\int_{\mathbb{R}^n}\left|A_j(x,k)\right|\left|\phi_k * h(x)\right|\,dx \\
&\quad\le \sum_{j\in\mathbb{N}}\lambda_j\left[\sum_{k\in\mathbb{Z}}\int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}}\left|A_j(x,k)\right|^2\,dx\right]^{\frac12} \\
&\qquad\times\left[\sum_{k\in\mathbb{Z}}\int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}}\left|\phi_k * h(x)\right|^2\,dx\right]^{\frac12} \\
&\quad= \sum_{j\in\mathbb{N}}\lambda_j\left[\sum_{k\in\mathbb{Z}}b^{-k}\int_{\{x\in\mathbb{R}^n:\,(x,k)\in\Gamma(y)\}}\left|A_j(x,k)\right|^2\,dx\int_{\{y\in\mathbb{R}^n:\,y\in x+B_k\}}dy\right]^{\frac12} \\
&\qquad\times\left[\sum_{k\in\mathbb{Z}}\int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}}\left|\phi_k * h(x)\right|^2\,dx\right]^{\frac12}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in\mathbb{N}} \lambda_j \left\|A_j\right\|_{T_2^{A,2}(\mathbb{R}^n\times\mathbb{Z})} \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} |\phi_k * h(x)|^2\,dx\right]^{\frac{1}{2}} \\
&\le \sum_{j\in\mathbb{N}} \frac{\lambda_j |x_j+B_{l_j}|^{\frac{1}{2}}}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} |\phi_k * h(x)|^2\,dx\right]^{\frac{1}{2}} \\
&\lesssim \|f\|_{H_X^A(\mathbb{R}^n)} \widetilde{\|d\mu\|}_X^{A,\theta_0},
\end{aligned}
$$

which, together with Theorem 3.15, Proposition 6.2, and Corollary 3.16, further implies that

$$
\|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \lesssim \|d\mu\|_X^{A,\theta_0}.
$$

This finishes the proof of (ii) and hence Theorem 6.3. □

7 Several applications

In this section, we apply Theorems 3.15, 4.1, 4.3, 5.4, 5.5, 5.6, and 6.3 as well as Corollary 3.16 to seven concrete examples of ball quasi-Banach function spaces, namely Morrey spaces (see Sect. 7.1 below), Orlicz-slice spaces (see Sect. 7.2 below), Lorentz spaces (see Sect. 7.3 below), variable Lebesgue spaces (see Sect. 7.4 below), mixed-norm Lebesgue spaces (see Sect. 7.5 below), weighted Lebesgue spaces (see Sect. 7.6 below), and Orlicz spaces (see Sect. 7.7 below).

7.1 Morrey spaces

Recall that the classical Morrey space $M_q^p(\mathbb{R}^n)$ with $0 < q \le p < \infty$, originally introduced by Morrey [66] in 1938, plays a fundamental role in harmonic analysis and partial differential equations. From then on, various variants of Morrey spaces over different underlying spaces have been investigated and developed (see, for instance, [17, 71]).

Definition 7.1 Let A be a dilation and $0 < q \le p < \infty$. The *anisotropic Morrey space* $M^p_{q,A}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$
\|f\|_{M^p_{q,A}(\mathbb{R}^n)} := \sup_{B\in\mathcal{B}} \left[|B|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(B)}\right] < \infty,
$$

where $\mathcal{B}$ is the same as in (2.4).

It is easy to show that $M^p_{q,A}(\mathbb{R}^n)$ is a ball quasi-Banach function space. From this and [84, Remark 8.4], we deduce that $M^p_{q,A}(\mathbb{R}^n)$ satisfies both Assumptions 2.10 and 2.12 with $X := M^p_{q,A}(\mathbb{R}^n)$, $p_- \in (0, q]$, $\theta_0 \in (0, \underline{p})$, and $p_0 \in (p, \infty)$, where $\underline{p} := \min\{p_-, 1\}$. In what follows, we always let $HM^p_{q,A}(\mathbb{R}^n)$ be the *anisotropic*

Hardy–Morrey space which is defined to be the same as in Definition 3.1 with $X := M^p_{q,A}(\mathbb{R}^n)$. Then, applying Theorems 5.4, 5.5, and 5.6, we obtain the following characterizations of $HM^p_{q,A}(\mathbb{R}^n)$, respectively, in terms of the anisotropic Lusin area function, the anisotropic Littlewood–Paley g-function, and the anisotropic Littlewood–Paley g^*_λ-function.

Theorem 7.2 *Let A be a dilation and $0 < q \le p < \infty$. Then Theorems 5.4, 5.5, and 5.6 with $X := M^p_{q,A}(\mathbb{R}^n)$ and $\lambda \in (2/\min\{1, q\}, \infty)$ hold true.*

Remark 7.3 (i) We point out that Theorem 7.2 is completely new.

(ii) However, Theorems 3.15, 4.1, 4.3, and 6.3 as well as Corollary 3.16 can not be applied to the anisotropic Morrey space $M^p_{q,A}(\mathbb{R}^n)$ because $M^p_{q,A}(\mathbb{R}^n)$ does not have an absolutely continuous quasi-norm.

7.2 Orlicz-Slice spaces

Recently, Zhang et al. [92] originally introduced the Orlicz-slice space on $\mathbb{R}^n$, which generalizes both the slice space in [2] and the Wiener-amalgam space in [25]. They also introduced the Orlicz-slice (local) Hardy spaces and developed a complete real-variable theory of these spaces in [91, 92]. For more studies about Orlicz-slice spaces, we refer the reader to [37, 38].

Recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called an *Orlicz function* if it is non-decreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0, \infty)$, and $\lim_{t\to\infty} \Phi(t) = \infty$. The function Φ is said to be of *upper* (resp. *lower*) *type* p for some $p \in [0, \infty)$ if there exists a positive constant C such that, for any $s \in [1, \infty)$ (resp. $s \in [0, 1]$) and $t \in [0, \infty)$, $\Phi(st) \le Cs^p\Phi(t)$. The *Orlicz space* $L^\Phi(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^\Phi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\} < \infty.$$

Definition 7.4 Let A be a dilation, $\ell \in \mathbb{Z}$, $q \in (0, \infty)$, and Φ be an Orlicz function. The *anisotropic Orlicz-slice space* $(E^q_\Phi)_{\ell,A}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{(E^q_\Phi)_{\ell,A}(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} \left[\frac{\|f\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}\right]^q dx\right\}^{\frac{1}{q}} < \infty,$$

where B_ℓ is the same as in (2.3).

Let A be a dilation, $\ell \in \mathbb{Z}$, $q \in (0, \infty)$, and Φ be an Orlicz function with positive lower type p^-_Φ and positive upper type p^+_Φ. Then, by the arguments similar to those used in the proofs of [92, Lemmas 2.28 and 4.5], we find that $(E^q_\Phi)_{\ell,A}(\mathbb{R}^n)$ is a ball quasi-Banach function space and has an absolutely continuous quasi-norm. From these and [84, Remark 8.14], we deduce that $(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ satisfies both Assumptions 2.10 and 2.12 with $X := (E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$, $p_- \in (0, \min\{p_\Phi^-, q\}]$, $\theta_0 \in (0, \underline{p})$, and $p_0 \in (\max\{p_\Phi^+, q\}, \infty)$, where $\underline{p} := \min\{p_-, 1\}$. In what follows, we always let $(HE_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ denote the *anisotropic Orlicz-slice Hardy space* which is defined to be the same as in Definition 3.1 with $X := (E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$. Moreover, by Theorems 3.15, 4.1, 4.3, 5.4, 5.5, 5.6, and 6.3 as well as Corollary 3.16 with X replaced by $(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$, we obtain the following conclusion.

Theorem 7.5 *Let A be a dilation, $\ell \in \mathbb{Z}$, $q \in (0, \infty)$, and Φ be an Orlicz function with positive lower type p_Φ^-. Then*

(i) *Theorems 3.15, 4.1, 4.3, and 6.3 as well as Corollary 3.16 with $X := (E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ hold true;*
(ii) *Theorems 5.4, 5.5, and 5.6 with $X := (E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ and $\lambda \in (\frac{2}{\min\{1, p_\Phi^-, q\}}, \infty)$ also hold true.*

Remark 7.6 We point out that Theorem 7.5 is completely new.

7.3 Lorentz spaces

Let $p \in (0, \infty]$ and $q \in (0, \infty]$. Recall that the *Lorentz space* $L^{p,q}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ with the following finite quasi-norm

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := \begin{cases} \left[\dfrac{q}{p}\displaystyle\int_0^\infty \left\{t^{\frac{1}{p}} f^*(t)\right\}^q \dfrac{dt}{t}\right]^{\frac{1}{q}} & \text{if } q \in (0, \infty), \\ \sup\limits_{t\in(0,\infty)} \left[t^{\frac{1}{p}} f^*(t)\right] & \text{if } q = \infty \end{cases}$$

with the usual modification made when $p = \infty$, where f^* denotes the *non-increasing rearrangement* of f, that is, for any $t \in (0, \infty)$,

$$f^*(t) := \left\{\alpha \in (0, \infty) : \ d_f(\alpha) \le t\right\}$$

with $d_f(\alpha) := |\{x \in \mathbb{R}^n : \ |f(x)| > \alpha\}|$ for any $\alpha \in (0, \infty)$.

Then, by [85, Remarks 2.7(ii), 4.21(ii), and 6.8(iv)], we conclude that $L^{p,q}(\mathbb{R}^n)$ satisfies all the assumptions of Definition 3.1 with $X := L^{p,q}(\mathbb{R}^n)$, $p_- \in (0, p]$, $\theta_0 \in (0, \underline{p})$, and $p_0 \in (p, \infty)$, where $\underline{p} := \min\{p_-, 1\}$, and that $L^{p,q}(\mathbb{R}^n)$ has an absolutely continuous quasi-norm. In what follows, we always let $H_A^{p,q}(\mathbb{R}^n)$ be the *anisotropic Hardy–Lorentz space* which is defined to be the same as in Definition 3.1 with $X := L^{p,q}(\mathbb{R}^n)$. By Theorems 3.15, 4.1, 4.3, 5.4, 5.5, 5.6, and 6.3 as well as Corollary 3.16 with X replaced by $L^{p,q}(\mathbb{R}^n)$, we obtain the following conclusion.

Theorem 7.7 *Let A be a dilation, $p \in (0, \infty)$, and $q \in (0, \infty]$. Then*

(i) *Theorems 3.15, 4.1, 4.3, and 6.3 as well as Corollary 3.16 with $X := L^{p,q}(\mathbb{R}^n)$ hold true;*

(ii) *Theorems* 5.4*,* 5.5*, and* 5.6 *with* $X := L^{p,q}(\mathbb{R}^n)$ *and* $\lambda \in (2/\min\{1, p\}, \infty)$ *also hold true.*

Remark 7.8 Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $q \in (0, \infty)$, where $C^{\log}(\mathbb{R}^n)$ is the same as in Sect. 7.4 below. We point out that Theorem 7.7(i) is a special case of [59, Theorems 1 and 2] with $p(\cdot) \equiv p \in (0, \infty)$ therein and that Theorem 7.7(ii) improves the corresponding results in [64, Theorems 2.7, 2.8, and 2.9] by widening the range of $p \in (0, 1]$ into $p \in (0, \infty)$. Although the variable Hardy–Lorentz space $L^{p(\cdot),q}(\mathbb{R}^n)$ is also a ball quasi-Banach function space, [59, Theorems 1 and 2] can not be deduced from Theorems 3.15 and 6.3. This is because the boundedness of the powered Hardy–Littlewood maximal operator on the associate space of $L^{p(\cdot),q}(\mathbb{R}^n)$ is still unknown, which makes it impossible to verify Assumption 2.12 with $X := L^{p(\cdot),q}(\mathbb{R}^n)$.

7.4 Variable Lebesgue spaces

Denote by $\mathcal{P}(\mathbb{R}^n)$ the *set of all the measurable functions* $p(\cdot)$ on $\mathbb{R}^n$ satisfying

$$0 < \widetilde{p_-} := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \le \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) =: \widetilde{p_+} < \infty. \tag{7.1}$$

For any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the *variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx < \infty,$$

equipped with the *quasi-norm* $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ defined by setting

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left[\frac{|f(x)|}{\lambda}\right]^{p(x)} dx \le 1\right\}.$$

Denote by $C^{\log}(\mathbb{R}^n)$ the set of all the functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the *globally log-Hölder continuous condition*, that is, there exist $C_{\log}(p), C_\infty \in (0, \infty)$ and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e + 1/|x - y|)}$$

and

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + \rho(x))}.$$

Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then, by [85, Remarks 2.7(iv), 4.21(v), and 6.8(vii)], we conclude that $L^{p(\cdot)}(\mathbb{R}^n)$ satisfies all the assumptions of Definition 3.1 with $X := L^{p(\cdot)}(\mathbb{R}^n)$, $p_- := \widetilde{p_-}$, $\theta_0 \in (0, \underline{\widetilde{p}})$, and $p_0 \in (\widetilde{p_+}, \infty]$, where $\widetilde{p_-}$ and $\widetilde{p_+}$ are the same as in (7.1) and $\underline{\widetilde{p}} := \min\{1, \widetilde{p_-}\}$, and that $L^{p(\cdot)}(\mathbb{R}^n)$ has an absolutely continuous quasi-norm. In what follows, we always let $H_A^{p(\cdot)}(\mathbb{R}^n)$ be the *anisotropic variable Hardy space* which is defined to be the same as in Definition 3.1 with $X := L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, by Theorems 3.15, 4.1, 4.3, 5.4, 5.5, 5.6, and 6.3 as well as Corollary 3.16 with X replaced by $L^{p(\cdot)}(\mathbb{R}^n)$, we obtain the following conclusion.

Theorem 7.9 *Let A be a dilation and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then*

(i) *Theorems 3.15, 4.1, 4.3, and 6.3 as well as Corollary 3.16 with $X := L^{p(\cdot)}(\mathbb{R}^n)$ hold true;*
(ii) *Theorems 5.4, 5.5, and 5.6 with $X := L^{p(\cdot)}(\mathbb{R}^n)$ and $\lambda \in (2/\min\{1, \widetilde{p}_-\}, \infty)$ also hold true, where $\widetilde{p_-}$ is the same as in (7.1).*

Remark 7.10 We point out that Theorem 7.9(i) was also obtained in [40, Theorems 1, 2, and 3, and Corollary 1] and Theorem 7.9(ii) improves the corresponding results in [60, Theorems 6.1, 6.2, and 6.3] by widening the range of $\lambda \in (1+2/\min\{2, \widetilde{p}_-\}, \infty)$ into $\lambda \in (2/\min\{1, \widetilde{p}_-\}, \infty)$.

7.5 Mixed-norm Lebesgue spaces

Let $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty]^n$. Recall that the *mixed-norm Lebesgue space* $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} \left\{\int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1}\, dx_1\right\}^{\frac{p_2}{p_1}} dx_2\right]^{\frac{p_3}{p_2}} \cdots dx_n\right\}^{\frac{1}{p_n}}$$

is finite with the usual modifications made when $p_i = \infty$ for some $i \in \{1, \ldots, n\}$.

Let $\vec{p} \in (0, \infty]^n$. Then, by both [90, p. 2047] and [39, Lemmas 3.2 and 4.4], we conclude that $L^{\vec{p}}(\mathbb{R}^n)$ satisfies all the assumptions of Definition 3.1 with $X := L^{\vec{p}}(\mathbb{R}^n)$, $p_- := \widehat{p_-}$, $\theta_0 \in (0, \underline{\widehat{p}})$, and $p_0 \in (\theta_0, \infty)$, where $\widehat{p_-} := \min\{p_1, \ldots, p_n\}$ and $\underline{\widehat{p}} := \min\{1, \widehat{p_-}\}$, and that $L^{\vec{p}}(\mathbb{R}^n)$ has an absolutely continuous quasi-norm. In what follows, we always let $H_A^{\vec{p}}(\mathbb{R}^n)$ be the *anisotropic mixed-norm Hardy space* which is defined to be the same as in Definition 3.1 with $X := L^{\vec{p}}(\mathbb{R}^n)$. Moreover, by Theorems 3.15, 4.1, 4.3, 5.4, 5.5, 5.6, and 6.3 as well as Corollary 3.16 with X replaced by $L^{\vec{p}}(\mathbb{R}^n)$, we obtain the following conclusion.

Theorem 7.11 *Let A be a dilation and $\vec{p} \in (0, \infty)^n$. Then*

(i) *Theorems 3.15, 4.1, 4.3, and 6.3 as well as Corollary 3.16 with $X := L^{\vec{p}}(\mathbb{R}^n)$ hold true;*
(ii) *Theorems 5.4, 5.5, and 5.6 with $X := L^{\vec{p}}(\mathbb{R}^n)$ and $\lambda \in (2/\min\{1, \widehat{p_-}\}, \infty)$ also hold true, where $\widehat{p_-} := \min\{p_1, \ldots, p_n\}$.*

Remark 7.12 (i) We point out that Theorem 7.11(i) was also obtained in [41, Theorems 3.4, 4.1, and 5.3 and Corollary 3.9] and Theorem 7.11(ii) improves the corresponding results in [39, Theorems 6.2, 6.3, and 6.4] by widening the range of $\lambda \in (1 + 2/\min\{2, \widehat{p_-}\}, \infty)$ into $\lambda \in (2/\min\{1, \widehat{p_-}\}, \infty)$.

(ii) Let $\vec{a} := (a_1, \ldots, a_n) \in [1, \infty]^n$. Then Theorem 7.11(i) with

$$A := \begin{pmatrix} 2^{a_1} & 0 & \cdots & 0 \\ 0 & 2^{a_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 2^{a_n} \end{pmatrix}$$

gives the dual space of the anisotropic mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ which was introduced in [19, Definition 3.3] and completely answers the open problem on the dual space of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ proposed in [19].

7.6 Weighted Lebesgue spaces

Let $p \in (0, \infty]$ and $w \in \mathcal{A}_\infty(A)$. From [85, Remarks 2.7(iii), 4.21(iii), and 6.8(v)], we deduce that $L_w^p(\mathbb{R}^n)$ satisfies all the assumptions of Definition 3.1 with $X := L_w^p(\mathbb{R}^n)$, $p_- \in (0, p/q_w]$, $\theta_0 \in (0, \min\{1, p_-\})$, and $p \in (\theta_0, \infty)$, where q_w is the same as in (5.10), and that $L_w^p(\mathbb{R}^n)$ has an absolutely continuous quasi-norm. In what follows, we always let $H_w^p(\mathbb{R}^n)$ be the *anisotropic weighted Hardy space* which is defined to be the same as in Definition 3.1 with $X := L_w^p(\mathbb{R}^n)$. By Theorems 3.15, 4.1, 4.3, 5.4, 5.5, 5.6, and 6.3 as well as Corollary 3.16 with X replaced by $L_w^p(\mathbb{R}^n)$, we obtain the following conclusion.

Theorem 7.13 *Let A be a dilation, $p \in (0, \infty)$, and $w \in \mathcal{A}_\infty(A)$. Then*

(i) *Theorems 3.15, 4.1, 4.3, and 6.3 as well as Corollary 3.16 with $X := L_w^p(\mathbb{R}^n)$ hold true;*
(ii) *Theorems 5.4, 5.5, and 5.6 with $X := L_w^p(\mathbb{R}^n)$ and $\lambda \in (2/\min\{1, q_w/p\}, \infty)$ also hold true, where q_w is the same as in (5.10).*

Remark 7.14 We point out that Theorem 7.13(i) is completely new and Theorem 7.13(ii) improves the corresponding results in [49, Theorems 2.14, 3.1, and 3.9] by widening the range of $p \in (0, 1]$ into $p \in (0, \infty)$.

7.7 Orlicz spaces

Let Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+. From [85, Remarks 2.7(iii), 4.21(iv), and 6.8(vi)], we deduce that $L^\Phi(\mathbb{R}^n)$ satisfies all the assumptions of Definition 3.1 with $X := L^\Phi(\mathbb{R}^n)$, $p_- \in (0, p_\Phi^-]$, $\theta_0 \in (0, \min\{p_\Phi^-, 1\})$, and $p_0 \in (\max\{p_\Phi^+, 1\}, \infty)$, and that $L^\Phi(\mathbb{R}^n)$ has an absolutely continuous quasi-norm. In what follows, we always let $H_A^\Phi(\mathbb{R}^n)$ be the *anisotropic Orlicz–Hardy space* which is defined to be the same as in Definition 3.1 with $X := L^\Phi(\mathbb{R}^n)$. Moreover, by Theorems 3.15, 4.1, 4.3, 5.4, 5.5, 5.6, and 6.3 as well as Corollary 3.16 with X replaced by $L^\Phi(\mathbb{R}^n)$, we obtain the following conclusion.

Theorem 7.15 *Let A be a dilation and Φ an Orlicz function with lower type $p_\Phi^- \in (0, \infty)$. Then*

(i) *Theorems 3.15, 4.1, 4.3, and 6.3 as well as Corollary 3.16 with* $X := L^{\Phi}(\mathbb{R}^n)$ *hold true;*
(ii) *Theorems 5.4, 5.5, and 5.6 with* $X := L^{\Phi}(\mathbb{R}^n)$ *and* $\lambda \in (2/\min\{1, p_{\Phi}^{-}\}, \infty)$ *also hold true.*

Remark 7.16 We point out that Theorem 7.15(i) is completely new and Theorem 7.15(ii) improves the corresponding results in [49, Theorems 2.14, 3.1, and 3.9] by widening the range of $p_{\Phi}^{-} \in (0, 1]$ into $p_{\Phi}^{-} \in (0, \infty)$.

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Declarations

Conflict of interest All authors state no conflict of interest.

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